PROBLEM 1. “STRETCHABLE” CAPACITOR

A plane-parallel capacitor has circular sides of area S initially separated by distance d. Each plate is of mass m and it can move horizontally without friction along a fixed, long, straight, perfectly conducting rod that passes through a small hole in the plate’s center. The rods meet at the center of the capacitor but are electrically insulated from each other by a piece of rubber of negligible thickness. The rods are very thin and their effect on the electric and magnetic fields inside (and outside) the capacitor is to be neglected. Each plate makes at all times a perfect electrical contact with the rod on which it sits. The plates are initially kept at rest and each one is charged with a positive electric charge Q. The capacitor is in vacuum; the vacuum permittivity ε_0 and the speed of light c are known. A constant voltage U is applied between the rods and the plates are let loose.

a. Express the magnitude of Q in terms of S, d, ε_0, and U so that the plates still remain at rest. Is this equilibrium stable or unstable? Please explain.

For the rest of the problem assume that the electric charge Q has the value determined above. At time $t = 0$ the plates are given a very small kick which takes them out of the equilibrium position, so that they start moving apart from each other. Denote the distance between the plates at some moment t by $x(t)$.

b. Write down the differential equation describing $x(t)$.

c. Transform the equation so that it describes the dependency of $v(x)$ (the velocity with which *the distance between the plates* varies). Solve the equation in terms of m, S, d, ε_0, U, and x.

d. Determine the rate with which the magnitude of the electric field E inside the capacitor varies in time.

Consider the points inside the capacitor situated at distance r from the axis passing through the centers of the plates, where r is much smaller than the radius of the plates.

e. Express the magnitude of the magnetic field B at these points in terms of m, S, d, ε_0, c, U, x, and r. How are the magnetic field lines oriented?

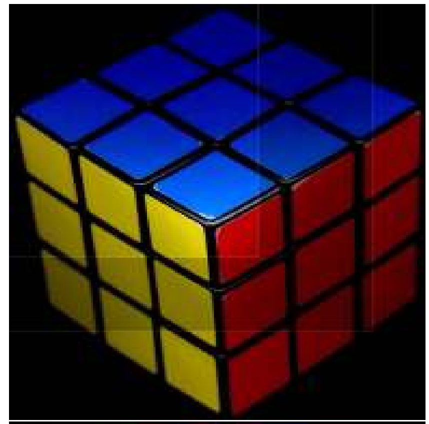
A small thin dielectric ring having area A, moment of inertia I, and positive electric charge q is placed inside the capacitor, at distance r from the axis passing through the centers of the plates. The plane of the ring includes the axis passing through the centers of the plates. Neglect at all times the inductances of all the objects in the problem.

At time $t = 0$ the ring is permitted to only rotate freely around its own axis.

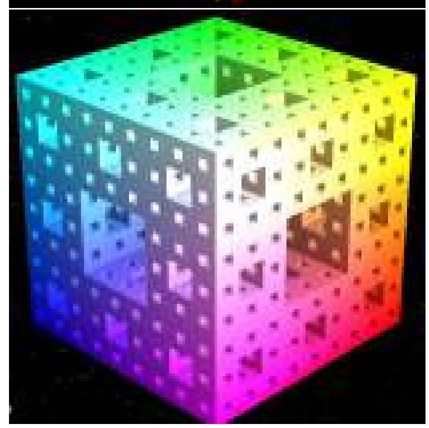
f. What will be the maximum angular velocity of the ring, and what will be the distance between the plates at that moment?

g. At the exact moment when the ring gets its maximal angular velocity, the voltage is disconnected. What will be the ring’s angular velocity right after that?

PROBLEM 2. FRACTAL PHYSICS



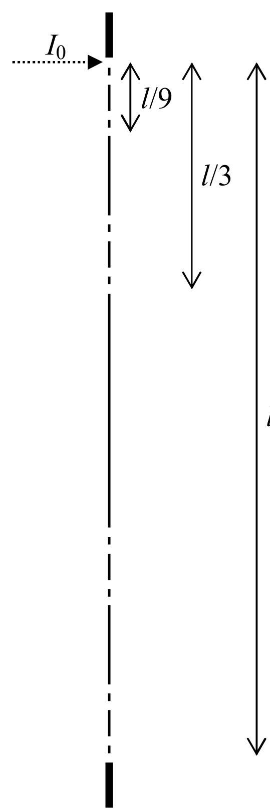
1. You are most probably familiar with the Rubik cube. It is a toy in the shape of a cube, having each side of a different color, made up of 9 smaller cubes, as in the picture alongside. So you have a total of 27 smaller cubes (the central one cannot be seen from outside, unless you break down the set).



Now remove this inner cube and the 6 cubes lying at the center of each side. You are left with 20 cubes, and now you can see along the axes passing through the centers of opposite sides.
Now apply the same procedure to each of these 20 cubes, and repeat it to infinity. In the end you get a fractal object looking like in the picture alongside, called the "Menger sponge" (or alternately, the "Sierpinski cube").

Calculate the moment of inertia of such an object having the resulting mass m and the length of the edge l, with respect to an axis passing trough the centers of two opposite sides.

2. Consider a one-dimensional diffraction grating in the shape of the so called "Cantor set". In order to obtain such a grating, start by blocking the central third of a narrow aperture of width l. Then you must block the central thirds of the two remaining apertures, and so on. The end result looks like in the picture alongside.



Let N be the maximum number of steps for which it is physically possible to apply this procedure, and I_0 the intensity of the monochromatic light falling normally onto each aperture of the grating. Consider the light diffracted at an angle α, and assume always the simplifying hypothesis that the apertures are point-like.

a. Write down the path difference corresponding to the two uppermost apertures, in terms of l, N and α. Calculate the intensity of the light diffracted by these two apertures as a function of α, ignoring all other apertures.

b. Let us denote $2\pi l \sin\alpha/3^N\lambda$ by x. Plot the graph of $I(x)$.

c. Similarly, calculate $I(\alpha)$ for the four uppermost apertures and plot $I(x)$.

[*Hint*: For the sake of simplicity, assume that the nontrivial solutions of the equation $\tan(x) + 3\tan(3x) = 0$ are $x = n\pi/6$, $n \in \mathbb{N}$.]

d. Express $I(\alpha)$ for the entire diffraction grating in terms of I_0, l, N, and α, and try to infer the general diffraction pattern.

PROBLEM 3. AN INTRODUCTION TO QUANTUM MECHANICS

In what follows the kinetic energy of a particle is supposed to be much smaller than its rest energy, so it can be written in a Newtonian manner:

$$E = E_0 + \frac{m_0 v^2}{2} = m_0 c^2 + \frac{p^2}{2m_0} = f(p) .$$

1. The equation of an undamped plane wave propagating in the positive direction of the x-axis is

$$\xi(x,t) = a\sin(\omega t - kx) ,$$

where the argument of the sine function is called the "phase" of the wave, and k is called the "wave number" and is equal to $2\pi/\lambda$.
The phase velocity c is defined as the velocity with which a certain value of the wave phase propagates in space, so strictly speaking it just represents the propagation velocity of the wave.

a. Express the phase velocity in terms of ω and k.

Now consider two undamped plane waves propagating in the positive direction of the x-axis with almost equal ω-s and k-s:

$$\xi_1(x,t) = a_1 \sin(\omega_1 t - k_1 x) \ , \ \xi_2(x,t) = a_2 \sin(\omega_2 t - k_2 x) \ ; \ \omega_1 \approx \omega_2 \ , \ k_1 \approx k_2 \ .$$

b. Taking into account the interference of these two waves at some moment of time t, determine the points on the x-axis corresponding to maxima of the amplitude of the resulting wave at that precise moment.

Let us define the group velocity v_g as the velocity with which these points move along the x-axis.

c. Express the group velocity in terms of $\Delta\omega = \omega_2 - \omega_1$ and $\Delta k = k_2 - k_1$.

Now instead of just two waves, consider a "packet" of waves having ω-s in a very narrow range $\Delta\omega$ around ω_0 and k-s in a very narrow range Δk around k_0. In this case one can think of ω as varying linearly with k, $\Delta\omega/\Delta k$ representing the gradient (slope). Also, one can speak of an almost constant amplitude density, $a/\Delta k$.

d. Write down the expression for the resulting wave in terms of a, ω_0, $\Delta\omega$, k_0, Δk, x, and t. For some moment of time t, estimate the ratio of the greatest two magnitudes of the amplitude. Also show that the expression already found for v_g still holds.

[*Hint*: The equation $\tan\alpha = \alpha$ has, besides the obvious solution $\alpha = 0$, the approximate solutions $\alpha = \pm(2n+1)\,\pi/2$, $n = 1, 2,\ldots$]

2. One knows that the energy and momentum of a photon can be expressed in terms of the frequency and/or the wavelength of the corresponding electromagnetic radiation:

$$E = h\nu = \frac{h}{2\pi} 2\pi\nu = \hbar\omega \; ; \; p = \frac{h}{\lambda} = \frac{2\pi}{\lambda}\frac{h}{2\pi} = \hbar k \; ; \; \hbar \approx 10^{-34}\,\text{Js}.$$

Louis DeBroglie put forth the hypothesis of extending the above formulas for all microscopic particles. Thus for any particle moving along the x-axis one can associate the equation of an undamped plane wave:

$$\xi(x,t) = a\sin\left(\frac{E}{\hbar}t - \frac{p}{\hbar}x\right).$$

Moreover, let us now assume that each particle is equivalent to a packet of waves centered on $E/\hbar$ and $p/\hbar$.

e. Show that in this context v_g is simply the particle's velocity.

The above results lead us to the idea that the main maximum of the packet could be interpreted as an indication regarding the position of the particle along the x-axis at some moment of time t.

f. Calculate the distance Δx between the center of the packet and the nearest minimum, and show that $\Delta x \Delta p = h$.

This means that if you want to have greater and greater accuracy for the position of the center of the packet, then you must allow for wider and wider ranges of the momentum of the particle. Alternately, as you try to reduce the wave packet to just one single wave corresponding to an exact value of the momentum, you loose track of the position of the particle along the axis.
At first sight, the distance calculated above could be interpreted as an indication regarding the relative size of the particle. But since in theory the width of the packet is arbitrary, and there are also other smaller maxima of the resulting wave, *it is far more interesting to view the resulting wave as a measure of the probability for finding the particle along the* x*-axis at some moment of time* t.

3. Starting from the real function of two real variables $\xi(x,t)$ one can naturally define a complex function of the same variables so that its imaginary part is the original function:

$$\Psi(x,t) = a[\cos(\omega t - kx) + i\sin(\omega t - kx)].$$

In what follows, we will use another notation for complex quantities, namely:

$$\Psi(x,t) = a\,\mathrm{e}^{i(\omega t - kx)}.$$

In what follows we will adopt this complex function as the expression for a plane wave.

g. Write down in the two notations the equation for an undamped plane wave propagating in the negative direction of the x-axis. Find out the expressions of the sine and cosine functions in exponential form.
h. Write down in exponential notation the equation for a standing wave obtained by interference of two undamped plane waves with equal amplitudes propagating in the two directions of the x-axis.

Let us define de probability density of finding a particle in a certain point x at some moment t by $|\Psi(x, t)|^2$. This means that

$$\int \Psi(x,t)\Psi^*(x,t)dx = 1 \quad \text{for any } t,$$

where the star (*) denotes the conjugate of a complex quantity.
We want to study the quantum states of a free moving particle (i.e. an electron having the rest energy $E_0 = 0.5$ MeV) in a one-dimensional region of length l. Consequently its potential energy is 0 for any $x \in [-\, l/2\ ,\ +\, l/2]$ and infinite otherwise. You can imagine this region as a one-dimensional well with infinitely high walls in zero gravity, while assuming that the interactions of the particle with the walls are perfectly elastic.

i. What is the lowest possible value for the kinetic energy of such a particle in classical mechanics? What is the probability density in this case?

The quantum approach of the problem resides in assimilating the particle to different standing waves (modes of oscillation) Ψ having nodes at $x = \pm\, l/2$, called "wave functions", *each mode of oscillation corresponding to a quantum state of the particle.*

j. Determine the values E_n for the kinetic energy of a particle corresponding to the n-th oscillation mode (n is called "quantum number"). Evaluate E_1 for an electron for $l = 1$ Å.
k. Write down the wave function Ψ_1 corresponding to the "ground (fundamental) state" and the wave function Ψ_2 corresponding to the first "excited state", and express their amplitudes c_1 and c_2 in terms of l. Evaluate the probabilities for finding the particle in the central third of the region in each of these two states.

Now just as the string of a musical instrument doesn't settle down for just one oscillation mode but rather is subject to a superposition of the possible standing waves, so does the real state of the particle consists of a mixture of all the wave functions, with some coefficients α_n representing the weights of each "pure" state:

$$\Psi(x,t) = \sum_{n=1}^{\infty} \alpha_n \Psi_n(x,t)\ .$$

l. Consider $\alpha_n = 0$ for all n > 2, and find out the relation which must exist in this case between α_1 and α_2.

In fact the above result holds for all n, and it shows that Ψ_n don't interfere with one another. Rather they act as independent unit-vectors of an infinite dimensional space, $\Psi(x,t)$ being basically a vector with coordinates $\{\alpha_n \mid n = 1, 2, \ldots\}$.

Finally let us revert to the assumption that there are only "pure" states and consider now the case of a particle moving freely inside a box with dimensions $l_x \times l_y \times l_z$. Again the potential energy is zero inside the box and infinite outside, and the interactions with the walls are perfectly elastic, but this time the wave function depends on four real variables: x, y, z, and t. This means that this time we will need three quantum numbers, one for each axis: n_x, n_y, and n_z.

m. What is the lowest possible value for the kinetic energy of such a particle in classical mechanics? What is the probability density in this case?
n. Write down the expressions for the kinetic energy of the particle in terms of E_0, c, l_x, l_y, l_z, n_x, n_y, n_z, and $\hbar$.

It can easily be seen that there can be situations in which the particle has the same kinetic energy for different sets of quantum numbers. This fact is called "degeneracy". A trivial condition which is sufficient to have degeneracy is that at least two sides of the box be equal.

o. For $l_x = l_y = l_z = 1$Å, evaluate the kinetic energy corresponding to the ground state of an electron. Is this state degenerate?
p. Find an example of degeneracy in the case that the sides of the box are not equal.
q. Write down the equation for $\Psi_{nx,ny,nz}(x,y,z,t)$ and determine the value of $c_{nx,ny,nz}$ in terms of l_x, l_y, and l_z.