a. **(1p)**
The electric charges of the plates are

$$q_{1,2} = Q \pm \frac{\varepsilon_0 SU}{d}$$

$$F = 0 \Rightarrow q = 0 \Rightarrow Q = \frac{\varepsilon_0 SU}{d}$$. The equilibrium is unstable.

b. **(1p)**

$$m\frac{\ddot{x}}{2} = \frac{q_1 q_2}{2\varepsilon_0 S} \Rightarrow \ddot{x} = \frac{\varepsilon_0 SU^2}{m}\left(\frac{1}{d^2} - \frac{1}{x^2}\right)$$

c. **(2p)**

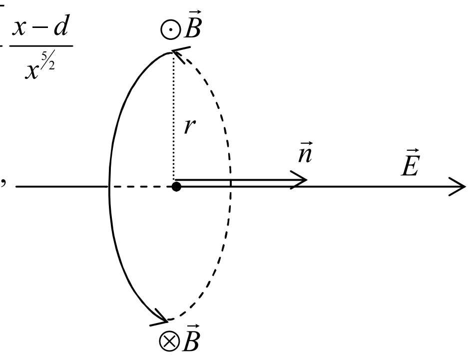
$$\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = \frac{\varepsilon_0 SU^2}{m}\left(\frac{\dot{x}}{d^2} - \frac{\dot{x}}{x^2}\right) \Rightarrow \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}t} = \frac{\varepsilon_0 SU^2}{m}\left(\frac{1}{d^2}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x}\right)\right) \Rightarrow \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}x} - \frac{\varepsilon_0 SU^2}{m}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = \frac{\varepsilon_0 SU^2}{m}\frac{1}{d^2}$$

$$\frac{m}{2\varepsilon_0 SU^2}\mathrm{d}(v^2) = \left(\frac{1}{d^2} - \frac{1}{x^2}\right)\mathrm{d}x \Rightarrow v^2 = \frac{2\varepsilon_0 SU^2}{m}\left(\frac{x}{d^2} + \frac{1}{x}\right) + C$$

$$x = d \Rightarrow v = 0 \Rightarrow C = -\frac{4\varepsilon_0 SU^2}{md} \Rightarrow v^2 = \frac{2\varepsilon_0 SU^2}{m}\left(\frac{\sqrt{x}}{d} - \frac{1}{\sqrt{x}}\right)^2 = \frac{2\varepsilon_0 SU^2}{md^2}\frac{(x-d)^2}{x}$$

d. **(1p)**

$$E = \frac{U}{x} \Rightarrow \dot{E} = -\frac{U}{x^2}\dot{x} = -\frac{U^2}{d}\sqrt{\frac{2\varepsilon_0 S}{m}}\frac{x-d}{x^{5/2}}$$



e. **(1p)**
For an arbitrarily chosen orientation of the magnetic lines,

$$B \cdot 2\pi r = \frac{1}{c^2}\dot{\Phi}_{\mathrm{el}} = \frac{1}{c^2}\pi r^2 \dot{E} \Rightarrow B(x) = -\frac{rU^2}{2c^2 d}\sqrt{\frac{2\varepsilon_0 S}{m}}\frac{x-d}{x^{5/2}}$$

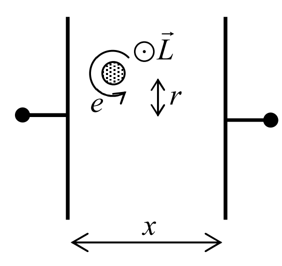
f. **(3p)**
The e.m.f. induced in the ring by the magnetic field is

$$e(x) = -\dot{\Phi}_{\mathrm{mag}} = -A\dot{B}.$$

Let ρ be the radius of the ring. The torque exerted upon the ring is

$$M(x) = F\rho = q\frac{e}{2\pi\rho}\rho = -\frac{qA}{2\pi}\dot{B}.$$

$$\dot{L} = I\dot{\omega} = -\frac{qA}{2\pi}\dot{B} \Rightarrow \omega(x) = -\frac{qAB}{2\pi I} + C.$$



$$x = d \Rightarrow B = 0 \Rightarrow C = 0 \Rightarrow \omega(x) = -\frac{qAB}{2\pi I} = \frac{qArU^2}{4\pi Ic^2 d}\sqrt{\frac{2\varepsilon_0 S}{m}}\frac{x-d}{x^{5/2}}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}x} = 0 \Rightarrow \frac{x^{5/2} - \frac{5}{2}x^{3/2}(x-d)}{x^{10}} = 0 \Rightarrow \frac{5}{2}d = \frac{3}{2}x \Rightarrow x = \frac{5}{3}d \Rightarrow \omega_{\max} = \frac{3qArU^2}{25\pi Ic^2 d^2}\sqrt{\frac{3\varepsilon_0 S}{10md}}$$

g. **(1p)**

$$x > d \Rightarrow E = \text{const.} \Rightarrow B = 0 \Rightarrow \omega = 0$$

1.

The mass of each of the 20 smaller cubes is $m_1 = m/20$. The length of the edge of each smaller cube is $l_1 = l/3$. So the moment of inertia of each smaller cube with respect to its own axis is $I_1 = I/180$. **(1p)**

There are 8 cubes whose axes are at a distance $l/3$ from the central axis. There are 12 cubes whose axes are at a distance $l\sqrt{2}/3$ from the central axis. According to the parallel axes theorem (Steiner)

$$I = 8\left(I_1 + \frac{m}{20}\frac{l^2}{9}\right) + 12\left(I_1 + \frac{m}{20}\frac{2l^2}{9}\right) \Rightarrow I = 20\frac{I}{20\cdot 9} + \frac{m}{20}\frac{l^2}{9}(8+24) \Rightarrow I = \frac{ml^2}{5} \quad \textbf{(3p)}$$

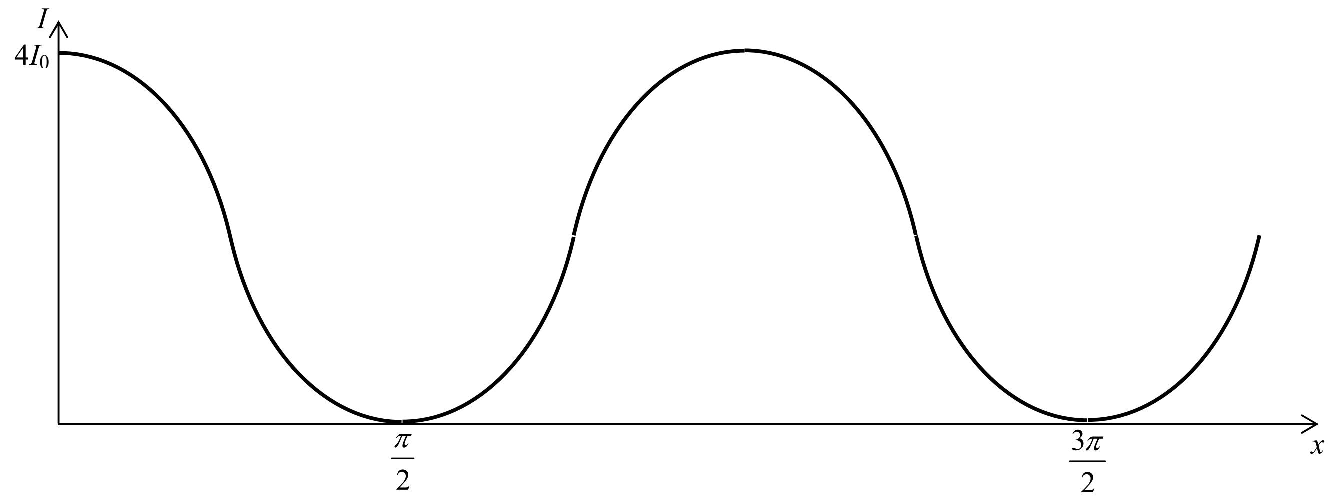
2.

a. (1p)

$$\delta = \frac{2l}{3^N}\sin\alpha$$

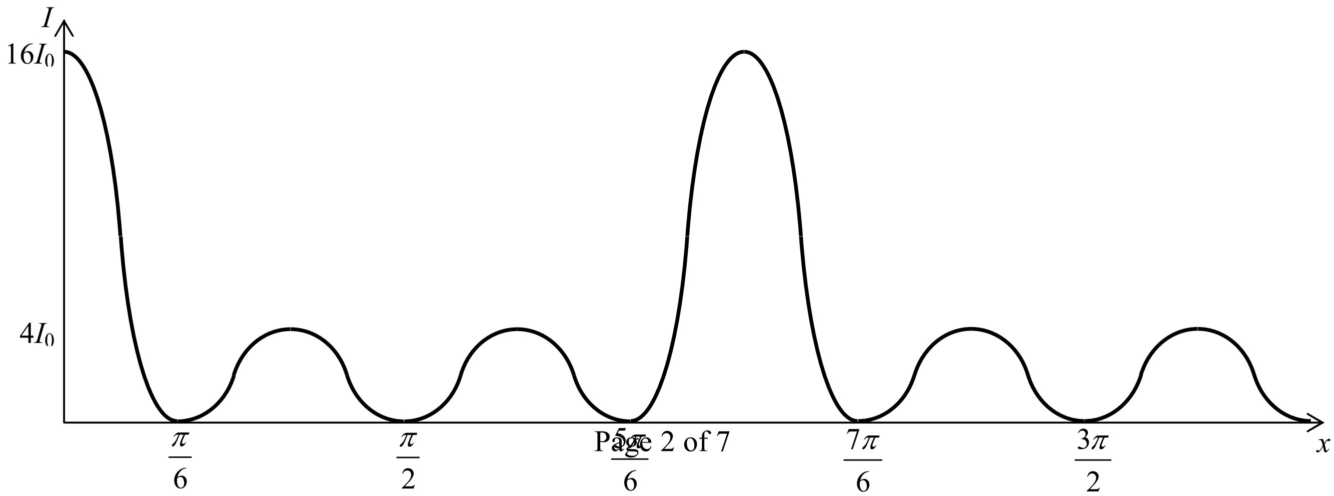
$$I(\alpha) = 2I_0\left(1+\cos\left(\frac{2\pi}{\lambda}\frac{2l}{3^N}\sin\alpha\right)\right) = 4I_0\cos^2\left(\frac{2\pi l}{3^N\lambda}\sin\alpha\right)$$

b. (1p)



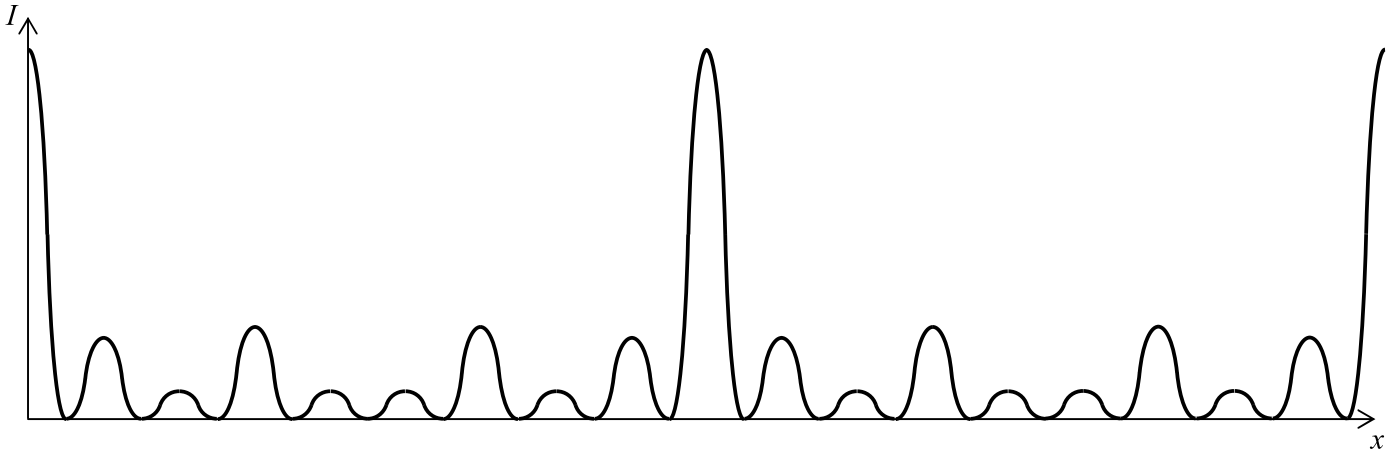
c. (1p) + (1p)

$$I(\alpha) = 8I_0\cos^2\left(\frac{2\pi l}{3^N\lambda}\sin\alpha\right)\left(1+\cos\left(\frac{2\pi}{\lambda}\frac{2l}{3^{N-1}}\sin\alpha\right)\right) = 16I_0\cos^2 x\cos^2 3x$$



d. (1p) + (1p)

$$I(\alpha) = 4^N I_0 \prod_{k=1}^{N} \cos^2\left(\frac{2\pi l}{3^k \lambda} \sin\alpha\right) = 4^N I_0 \prod_{k=1}^{N} \cos^2 3^{k-1} x$$



So the diffraction pattern mimics perfectly the fractal character of the diffraction grating.

1.

a. (0.5p)

$$\omega t - kx = \text{const.} \Rightarrow \omega\,\mathrm{d}\,t - k\,\mathrm{d}\,x = 0 \Rightarrow c = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{\omega}{k}$$

b. (0.5p)

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\left(\omega_1 t - k_1 x - \omega_2 t + k_2 x\right)}$$

$$\cos\left(\left(\omega_1 - \omega_2\right)t - \left(k_1 - k_2\right)x\right) = 1 \Rightarrow \Delta\omega t - \Delta k x = 2n\pi \ ,\ n \in \mathbb{Z}$$

c. (0.5p)

$$v_g = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{\Delta\omega}{\Delta k}$$

d. (1.5p)

$$\omega(k) = \omega_0 + \frac{\Delta\omega}{\Delta k}\left(k - k_0\right) \Rightarrow \mathrm{d}\,\xi(x,t) = \mathrm{d}\,a\sin\left(\left(\omega_0 + \frac{\Delta\omega}{\Delta k}\left(k - k_0\right)\right)t - kx\right)$$

$$\xi(x,t) = \int_{k_0 - \frac{\Delta k}{2}}^{k + \frac{\Delta k}{2}} \frac{a}{\Delta k}\sin\left(\left(\omega_0 - \frac{\Delta\omega}{\Delta k}k_0\right)t + \left(\frac{\Delta\omega}{\Delta k}t - x\right)k\right)\mathrm{d}\,k$$

$$\xi(x,t) = -\frac{a}{\Delta k}\left.\frac{\cos\left(\left(\omega_0 - \frac{\Delta\omega}{\Delta k}k_0\right)t + \left(\frac{\Delta\omega}{\Delta k}t - x\right)k\right)}{\frac{\Delta\omega}{\Delta k}t - x}\right|_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}}$$

$$\xi(x,t) = 2a\frac{\sin\left(\frac{\Delta\omega t - \Delta k x}{2}\right)}{\Delta\omega t - \Delta k x}\sin\left(\omega_0 t - k_0 x\right)$$

$$\frac{\sin\left(\frac{\Delta\omega t - \Delta k x}{2}\right)}{\Delta\omega t - \Delta k x} = \text{const.} \Rightarrow \Delta\omega t - \Delta k x = \text{const.} \Rightarrow v_g = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{\Delta\omega}{\Delta k}$$

Viewing the expression above as a function of *x*, the points on the *x*-axis characterized by maxima of the amplitude at some moment of time *t* satisfy the condition

$$f'(x) = 0 \Rightarrow \frac{-\Delta k\left[\left(\Delta\omega t - \Delta k x\right)\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)\frac{1}{2} - \sin\left(\frac{\Delta\omega t - \Delta k x}{2}\right)\right]}{\left(\Delta\omega t - \Delta k x\right)^2} = 0$$

$$\tan\left(\frac{\Delta\omega t - \Delta k x}{2}\right) = \frac{\Delta\omega t - \Delta k x}{2} \Rightarrow \frac{\Delta\omega t - \Delta k x}{2} \approx \pm(2n+1)\frac{\pi}{2} \ ,\ n \in \mathbb{N}^*$$

$$\Delta\omega t - \Delta k x = 0 \Rightarrow x = \frac{\Delta\omega t}{\Delta k} \Rightarrow f\left(\frac{\Delta\omega t}{\Delta k}\right) = \frac{1}{2}$$

$$\Delta\omega t - \Delta k x = 3\pi \Rightarrow x = \frac{\Delta\omega t - 3\pi}{\Delta k} \Rightarrow f\left(\frac{\Delta\omega t - 3\pi}{\Delta k}\right) = \frac{\sin\left(\frac{3\pi}{2}\right)}{3\pi} = -\frac{1}{3\pi}$$

$$\left|\frac{f\left(\frac{\Delta\omega t - 3\pi}{\Delta k}\right)}{f\left(\frac{\Delta\omega t}{\Delta k}\right)}\right| = \frac{2}{3\pi} \approx 0.2$$

2.

e. (1p)

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}\omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}k} = \frac{\mathrm{d}\omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}k} = \frac{1}{\hbar}v\hbar = v$$

f. (1p)

$$\sin\left(\frac{\Delta\omega t - \Delta k x}{2}\right) = 0 \Rightarrow \frac{\Delta\omega t - \Delta k x}{2} = n\pi \ ,\ n \in \mathbb{N}^* \Rightarrow x = \frac{\Delta\omega t - 2\pi}{\Delta k}$$

$$\Delta x = \frac{\Delta\omega t - 2\pi}{\Delta k} - \frac{\Delta\omega t}{\Delta k} = \frac{2\pi}{\Delta k} = \frac{2\pi}{\frac{\Delta p}{\hbar}} \Rightarrow \Delta x \Delta p = \frac{h}{\pi}\pi = h$$

g. (1p)

$$\Psi(x,t) = a\left[\cos\left(\omega t + kx\right) + i\sin\left(\omega t + kx\right)\right] = a\,\mathrm{e}^{i(\omega t + kx)}$$

$$\sin\left(\omega t + kx\right) = \frac{\mathrm{e}^{i(\omega t + kx)} - \mathrm{e}^{-i(\omega t + kx)}}{2i}$$

$$\cos\left(\omega t + kx\right) = \frac{\mathrm{e}^{i(\omega t + kx)} + \mathrm{e}^{-i(\omega t + kx)}}{2}$$

h. (0.5p)

$$\Psi(x,t) = a\,\mathrm{e}^{i(\omega t - kx)} + a\,\mathrm{e}^{i(\omega t + kx)} = a\,\mathrm{e}^{i\omega t}\,2\frac{\mathrm{e}^{-ikx} + \mathrm{e}^{ikx}}{2} = 2a\cos kx\,\mathrm{e}^{i\omega t}$$

i. (0.25p)

$$E = 0\ ;\ \left|\Psi(x,t)\right|^2 = \frac{1}{l}$$

j. (0.5p)

$$l = n\frac{\lambda}{2}\ ;\ n \in \mathbb{N}^* \Rightarrow \lambda_n = \frac{2l}{n} \Rightarrow p_n = \frac{h}{\lambda_n} = n\frac{\pi\hbar}{l} \Rightarrow E_n = \frac{p_n^2}{2m} = \frac{\pi^2\hbar^2}{2ml^2}n^2$$

$$E_1 = \frac{\pi^2\hbar^2}{2ml^2} = \frac{\left(\pi\hbar c\right)^2}{2E_0 l^2} = \frac{\left(3.14\cdot 10^{-34}\,\mathrm{Js}\cdot 3\cdot 10^{8}\,\mathrm{m/s}\right)^2}{2\cdot 0.5\mathrm{MeV}\cdot 10^{-20}\,\mathrm{m}^2} = 34.81\mathrm{eV}$$

k. **(0.5p)**

$$\Psi_1(x,t) = c_1 \cos k_1 x \, \mathrm{e}^{i\omega_1 t} = c_1 \cos\left(\frac{\pi}{l}x\right)\mathrm{e}^{i\omega_1 t}$$

$$\Psi_2(x,t) = c_2 \sin k_2 x \, \mathrm{e}^{i\omega_2 t} = c_2 \sin\left(\frac{2\pi}{l}x\right)\mathrm{e}^{i\omega_2 t}$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \left|\Psi_1(x,t)\right|^2 \mathrm{d}\,x = 1 \Rightarrow c_1^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \cos^2\left(\frac{\pi}{l}x\right)\mathrm{d}\,x = 1 \Rightarrow c_1 = \sqrt{\frac{2}{l}}$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \left|\Psi_2(x,t)\right|^2 \mathrm{d}\,x = 1 \Rightarrow c_2^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \sin^2\left(\frac{2\pi x}{l}\right)\mathrm{d}\,x = 1 \Rightarrow c_2 = \sqrt{\frac{2}{l}}$$

$$P_1\left(\left[-\tfrac{l}{6},+\tfrac{l}{6}\right]\right) = \int_{-\frac{l}{6}}^{\frac{l}{6}} \left|\Psi_1(x,t)\right|^2 \mathrm{d}\,x = \frac{2}{l}\int_{-\frac{l}{6}}^{\frac{l}{6}} \frac{1+\cos\left(\frac{2\pi x}{l}\right)}{2}\mathrm{d}\,x = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}$$

$$P_2\left(\left[-\tfrac{l}{6},+\tfrac{l}{6}\right]\right) = \int_{-\frac{l}{6}}^{\frac{l}{6}} \left|\Psi_2(x,t)\right|^2 \mathrm{d}\,x = \frac{2}{l}\int_{-\frac{l}{6}}^{\frac{l}{6}} \frac{1-\cos\left(\frac{4\pi x}{l}\right)}{2}\mathrm{d}\,x = \frac{1}{3} - \frac{\sqrt{3}}{4\pi}$$

l. **(0.5p)**

$\Psi(x,t) = \alpha_1 c_1 \cos k_1 x \, \mathrm{e}^{i\omega_1 t} + \alpha_2 c_2 \sin k_2 x \, \mathrm{e}^{i\omega_2 t}$; $\Psi^*(x,t) = \alpha_1 c_1 \cos k_1 x \, \mathrm{e}^{-i\omega_1 t} + \alpha_2 c_2 \sin k_2 x \, \mathrm{e}^{-i\omega_2 t}$

$$\left|\Psi(x,t)\right|^2 = \alpha_1^2 c_1^2 \cos^2 k_1 x + \alpha_2^2 c_2^2 \sin^2 k_2 x + \alpha_1\alpha_2 c_1 c_2 \cos k_1 x \sin k_2 x\left(\mathrm{e}^{i(\omega_1-\omega_2)t} + \mathrm{e}^{i(\omega_2-\omega_1)t}\right)$$

$$\left|\Psi(x,t)\right|^2 = \frac{2}{l}\left[\alpha_1^2 \cos^2 k_1 x + \alpha_2^2 \sin k_2 x + 2\alpha_1\alpha_2 \cos k_2 x \sin k_2 x \cos\left(\left(\omega_1-\omega_2\right)t\right)\right]$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \left|\Psi(x,t)\right|^2 \mathrm{d}x = 1 \Rightarrow \alpha_1^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1+\cos 2k_1 x}{2}\mathrm{d}\,x + \alpha_2^2 \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1-\cos 2k_2 x}{2}\mathrm{d}\,x +$$

$$+\alpha_1\alpha_2 \cos\left(\left(\omega_1-\omega_2\right)t\right)\int_{-\frac{l}{2}}^{\frac{l}{2}}\left[\sin\left(\left(k_2+k_1\right)t\right) - \sin\left(\left(k_2-k_1\right)t\right)\right]\mathrm{d}\,x = \frac{l}{2}$$

$$\alpha_1^2 + \alpha_2^2 = 1$$

m. **(0.25p)**

$$E = 0 \;;\; \left|\Psi(x,y,z,t)\right|^2 = \frac{1}{l_x l_y l_z}$$

n. **(0.25p)**

The movement of the particle on each axis is independent, so

$$E_{n_x,n_y,n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \frac{\pi^2\hbar^2 c^2}{2E_0}\left(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2}\right)$$

o. **(0.25p)**

$E_{1,1,1} = 3E_1 \approx 104.43\mathrm{eV}$. The ground state is not degenerate.

p. **(0.5p)**

$$l_2 = 2l_1 \;\; ; \;\; n_x = 5 \text{ and } n_y = 8 \;\; ; \;\; n_x' = 4 \text{ and } n_y' = 10$$

q. **(0.5p)**

$$\Psi_{n_x,n_y,n_z}(x,y,z,t) = c_{n_x,n_y,n_z} \sin\left(k_{n_x} x + n_x \frac{\pi}{2}\right) \sin\left(k_{n_y} y + n_y \frac{\pi}{2}\right) \sin\left(k_{n_z} z + n_z \frac{\pi}{2}\right) \mathrm{e}^{i\left(\omega_{n_x}+\omega_{n_y}+\omega_{n_z}\right)t}$$

$$\int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} \int_{-\frac{l_y}{2}}^{\frac{l_y}{2}} \int_{-\frac{l_z}{2}}^{\frac{l_z}{2}} \left|\Psi_{n_x,n_y,n_z}(x,y,x,t)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y\,\mathrm{d}\,z = 1$$

$$c^2_{n_x,n_y,n_z} \int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} \frac{1 \pm \cos\left(2k_{n_x} x\right)}{2} \mathrm{d}\,x \int_{-\frac{l_y}{2}}^{\frac{l_y}{2}} \frac{1 \pm \cos\left(2k_{n_y} y\right)}{2} \mathrm{d}\,y \int_{-\frac{l_z}{2}}^{\frac{l_z}{2}} \frac{1 \pm \cos\left(2k_{n_z} z\right)}{2} \mathrm{d}\,z = 1$$

$$c^2_{n_x,n_y,n_z} \frac{l_x}{2} \frac{l_y}{2} \frac{l_z}{2} = 1 \Rightarrow c_{n_x,n_y,n_z} = \sqrt{\frac{8}{l_x l_y l_z}}$$