The 4$^{\text{th}}$ Romanian Master of Mathematics Competition – Solutions
Day 1: Friday, February 25, 2011, Bucharest

Problem 1. Prove that there exist two functions

$$f, g\colon \mathbb{R} \to \mathbb{R},$$

such that $f \circ g$ is strictly decreasing, while $g \circ f$ is strictly increasing.

(Poland) Andrzej Komisarski & Marcin Kuczma

Solution. Let

- $A = \bigcup_{k\in\mathbb{Z}} \left(\left[-2^{2k+1}, -2^{2k} \right) \bigcup \left(2^{2k}, 2^{2k+1} \right] \right);$
- $B = \bigcup_{k\in\mathbb{Z}} \left(\left[-2^{2k}, -2^{2k-1} \right) \bigcup \left(2^{2k-1}, 2^{2k} \right] \right).$

Thus $A = 2B$, $B = 2A$, $A = -A$, $B = -B$, $A \cap B = \emptyset$, and finally $A \cup B \cup \{0\} = \mathbb{R}$. Let us take

$$f(x) = \begin{cases} x & \text{for} \quad x \in A; \\ -x & \text{for} \quad x \in B; \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Take $g(x) = 2f(x)$. Thus $f(g(x)) = f(2f(x)) = -2x$ and $g(f(x)) = 2f(f(x)) = 2x$. ■

Problem 2. Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;

(2) the degree of f is less than n.

(Hungary) Géza Kós

Solution. We will show that such polynomial exists if and only if $n = 1$ or n is a power of a prime.

We will use two known facts stated in Lemmata 1 and 2.

Lemma 1. If p^a is a power of a prime and k is an integer, then $\dfrac{(k-1)(k-2)\dots(k-p^a+1)}{(p^a-1)!}$ is divisible by p if and only if k is not divisible by p^a.

Proof. First suppose that $p^a \mid k$ and consider

$$\frac{(k-1)(k-2)\cdots(k-p^a+1)}{(p^a-1)!} = \frac{k-1}{p^a-1} \cdot \frac{k-2}{p^a-2} \cdots \frac{k-p^a+1}{1}.$$

In every fraction on the right-hand side, p has the same maximal exponent in the numerator as in the denominator. Therefore, the product (which is an integer) is not divisible by p.

Now suppose that $p^a \nmid k$. We have

$$\frac{(k-1)(k-2)\cdots(k-p^a+1)}{(p^a-1)!} = \frac{p^a}{k} \cdot \frac{k(k-1)\cdots(k-p^a+1)}{(p^a)!}.$$

The last fraction is an integer. In the fraction $\frac{p^a}{k}$, the denominator k is not divisible by p^a. □

Lemma 2. If $g(x)$ is a polynomial with degree less than n then

$$\sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} g(x+n-\ell) = 0.$$

Proof. Apply induction on n. For $n = 1$ then $g(x)$ is a constant and

$$\binom{1}{0} g(x+1) - \binom{1}{1} g(x) = g(x+1) - g(x) = 0.$$

Now assume that $n > 1$ and the Lemma holds for $n-1$. Let $h(x) = g(x+1) - g(x)$; the degree of h is less than the degree of g, so the induction hypothesis applies for g and $n-1$:

$$\sum_{\ell=0}^{n-1} (-1)^\ell \binom{n-1}{\ell} h(x+n-1-\ell) = 0$$

$$\sum_{\ell=0}^{n-1} (-1)^\ell \binom{n-1}{\ell} \big(g(x+n-\ell) - g(x+n-1-\ell) \big) = 0$$

$$\binom{n-1}{0} g(x+n) + \sum_{\ell=1}^{n-1} (-1)^\ell \left(\binom{n-1}{\ell-1} + \binom{n-1}{\ell} \right) g(x+n-\ell) - (-1)^{n-1} \binom{n-1}{n-1} g(x) = 0$$

$$\sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} g(x+n-\ell) = 0.$$

□

Lemma 3. If n has at least two distinct prime divisors then the greatest common divisor of $\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}$ is 1.

Proof. Suppose to the contrary that p is a common prime divisor of $\binom{n}{1}, \dots, \binom{n}{n-1}$. In particular, $p \mid \binom{n}{1} = n$. Let a be the exponent of p in the prime factorization of n. Since n has at least two prime divisors, we have $1 < p^a < n$. Hence, $\binom{n}{p^a-1}$ and $\binom{n}{p^a}$ are listed among $\binom{n}{1}, \dots, \binom{n}{n-1}$ and thus $p \mid \binom{n}{p^a}$ and $p \mid \binom{n}{p^a-1}$. But then p divides $\binom{n}{p^a} - \binom{n}{p^a-1} = \binom{n-1}{p^a-1}$, which contradicts Lemma 1. □

Next we construct the polynomial $f(x)$ when $n=1$ or n is a power of a prime.

For $n=1$, $f(x)=\frac{1}{2}$ is such a polynomial.

If $n=p^a$ where p is a prime and a is a positive integer then let

$$f(x)=\frac{1}{p}\binom{x-1}{p^a-1}=\frac{1}{p}\cdot\frac{(x-1)(x-2)\cdots(x-p^a+1)}{(p^a-1)!}.$$

The degree of this polynomial is $p^a-1=n-1$.

The number $\frac{(k-1)(k-2)\cdots(k-p^a+1)}{(p^a-1)!}$ is an integer for any integer k, and, by Lemma 1, it is divisible by p if and only if k is not divisible by $p^a=n$.

Finally we prove that if n has at least two prime divisors then no polynomial $f(x)$ satisfies (1,2). Suppose that some polynomial $f(x)$ satisfies (1,2), and apply Lemma 2 for $g=f$ and $x=-k$ where $1\le k\le n-1$. We get that

$$\binom{n}{k}f(0)=\sum_{0\le\ell\le n,\,\ell\ne k}(-1)^{k-\ell}\binom{n}{\ell}f(-k+\ell).$$

Since $f(-k),\dots,f(-1)$ and $f(1),\dots,f(n-k)$ are all integers, we conclude that $\binom{n}{k}f(0)$ is an integer for every $1\le k\le n-1$.

By dint of Lemma 3, the greatest common divisor of $\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n-1}$ is 1. Hence, there will exist some integers $u_1,u_2,\dots,u_{n-1}$ for which $u_1\binom{n}{1}+\dots+u_{n-1}\binom{n}{n-1}=1$. Then

$$f(0)=\left(\sum_{k=1}^{n-1}u_k\binom{n}{k}\right)f(0)=\sum_{k=1}^{n-1}u_k\binom{n}{k}f(0)$$

is a sum of integers. This contradicts the fact that $f(0)$ is not an integer. So such polynomial $f(x)$ does not exist. ■

Alternative Solution. (I. Bogdanov) We claim the answer is $n=p^\alpha$ for some prime p and nonnegative α.

Lemma. For every integers $a_1,\dots,a_n$ there exists an integer-valued polynomial $P(x)$ of degree $<n$ such that $P(k)=a_k$ for all $1\le k\le n$.

Proof. Induction on n. For the base case $n=1$ one may set $P(x)=a_1$. For the induction step, suppose that the polynomial $P_1(x)$ satisfies the desired property for all $1\le k\le n-1$. Then set $P(x)=P_1(x)+(a_n-P_1(n))\binom{x-1}{n-1}$; since $\binom{k-1}{n-1}=0$ for $1\le k\le n-1$ and $\binom{n-1}{n-1}=1$, the polynomial $P(x)$ is a sought one. □

Now, if for some n there exists some polynomial $f(x)$ satisfying the problem conditions, one may choose some integer-valued polynomial $P(x)$ (of degree $<n-1$) coinciding with $f(x)$ at points $1,\dots,n-1$. The difference $f_1(x)=f(x)-P(x)$ also satisfies the problem conditions, therefore we may restrict ourselves to the polynomials vanishing at points $1,\dots,n-1$ — that are, the polynomials of the form $f(x)=c\prod_{i=1}^{n-1}(x-i)$ for some (surely rational) constant c. Let $c=p/q$ be its irreducible form, and $q=\prod_{j=1}^{d}p_j^{\alpha_j}$ be the prime decomposition of the denominator.

1. Assume that a desired polynomial $f(x)$ exists. Since $f(0)$ is not an integer, we have $q\nmid(-1)^{n-1}(n-1)!$ and hence $p_j^{\alpha_j}\nmid(-1)^{n-1}(n-1)!$ for some j. Hence

$$\prod_{i=1}^{n-1}(p_j^{\alpha_j}-i)\equiv(-1)^{n-1}(n-1)!\not\equiv 0\pmod{p_j^{\alpha_j}},$$

therefore $f(p_i^{\alpha_i})$ is not integer, too. By the condition (i), this means that $n\mid p_i^{\alpha_i}$, and hence n should be a power of a prime.

2. Now let us construct a desired polynomial $f(x)$ for any power of a prime $n=p^\alpha$. We claim that the polynomial

$$f(x)=\frac{1}{p}\binom{x-1}{n-1}=\frac{n}{px}\binom{x}{n}$$

fits. Actually, consider some integer x. From the first representation, the denominator of the irreducible form of $f(x)$ may be 1 or p only. If $p^\alpha\nmid x$, then the prime decomposition of the fraction $n/(px)$ contains p with a nonnegative exponent; hence $f(x)$ is integer. On the other hand, if $n=p^\alpha\mid x$, then the numbers $x-1,x-2,\dots,x-(n-1)$ contain the same exponents of primes as the numbers $n-1,n-2,\dots,1$ respectively; hence the number

$$\binom{x-1}{n-1}=\frac{\prod_{i=1}^{n-1}(x-i)}{\prod_{i=1}^{n-1}(n-i)}$$

is not divisible by p. Thus $f(x)$ is not an integer. ■

Problem 3. A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

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Solution. Let P be the meeting point of the common inner tangents to γ_1 and γ_2. Also, let b be the angle bisector of $\angle BAC$. Since $KL\parallel BC$, b is also the angle bisector of $\angle KAL$.

Let $\mathfrak{H}$ be the composition of the symmetry $\mathfrak{S}$ with respect to b and the inversion $\mathfrak{I}$ of centre A and ratio $\sqrt{AK\cdot AL}$ (it is readily seen that $\mathfrak{S}$ and $\mathfrak{I}$ commute, so since $\mathfrak{S}^2=\mathfrak{I}^2=\mathrm{id}$, then also $\mathfrak{H}^2=\mathrm{id}$, the identical transformation). The elements of the configuration interchanged by $\mathfrak{H}$ are summarized in Table I.

Let O_1 and O_2 be the centres of circles γ_1 and γ_2. Since the circles γ_1 and γ_2 are determined by their construction (in a unique way), they are interchanged by $\mathfrak{H}$, therefore the rays AO_1 and AO_2 are symmetrical with respect to b. Denote by ρ_1 and ρ_2 the radii of γ_1 and γ_2. Since $\angle O_1AB = \angle O_2AC$, we have $\rho_1/\rho_2 = AO_1/AO_2$. On the other hand, from the definition of P we have $O_1P/O_2P = \rho_1/\rho_2 = AO_1/AO_2$; this means that AP is the angle bisector of $\angle O_1AO_2$ and therefore of $\angle BAC$.

The limiting, degenerated, cases are when the parallel line passes through A – when P coincides with A; respectively when the parallel line is BC – when P coincides with the foot $A' \in BC$ of the angle bisector of $\angle BAC$ (or any other point on BC). By continuity, any point P on the open segment AA' is obtained for some position of the parallel, therefore the locus is the open segment AA' of the angle bisector b of $\angle BAC$. ■

point K	$\longleftrightarrow$	point L
line KL	$\longleftrightarrow$	circle ω
ray AB	$\longleftrightarrow$	ray AC
point B	$\longleftrightarrow$	point E
point C	$\longleftrightarrow$	point D
segment BD	$\longleftrightarrow$	segment EC
arc BK	$\longleftrightarrow$	segment EL
arc CL	$\longleftrightarrow$	segment DK

TABLE I: Elements interchanged by $\mathfrak{H}$.