The 5^{th} Romanian Master of Mathematics Competition

Solutions for the Day 1

Problem 1. Given a finite group of boys and girls, a *covering set of boys* is a set of boys such that every girl knows at least one boy in that set; and a *covering set of girls* is a set of girls such that every boy knows at least one girl in that set. Prove that the number of covering sets of boys and the number of covering sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

Solution 1. A set X of boys is *separated* from a set Y of girls if no boy in X is an acquaintance of a girl in Y. Similarly, a set Y of girls is *separated* from a set X of boys if no girl in Y is an acquaintance of a boy in X. Since acquaintance is assumed mutual, separation is symmetric: X is separated from Y if and only if Y is separated from X.

This enables doubly counting the number n of ordered pairs (X, Y) of separated sets X, of boys, and Y, of girls, and thereby showing that it is congruent modulo 2 to both numbers in question.

Given a set X of boys, let Y_X be the largest set of girls separated from X, to deduce that X is separated from exactly $2^{|Y_X|}$ sets of girls. Consequently, $n = \sum_X 2^{|Y_X|}$ which is clearly congruent modulo 2 to the number of covering sets of boys.

Mutatis mutandis, the argument applies to show n congruent modulo 2 to the number of covering sets of girls.

Remark. The argument in this solution translates verbatim in terms of the adjancency matrix of the associated acquaintance graph.

Solution 2. (Ilya Bogdanov) Let B denote the set of boys, let G denote the set of girls and induct on $|B| + |G|$. The assertion is vacuously true if either set is empty.

Next, fix a boy b, let $B' = B \setminus \{b\}$, and let G' be the set of all girls who do not know b. Notice that:

(1) a covering set of boys in $B' \cup G$ is still one in $B \cup G$; and

(2) a covering set of boys in $B \cup G$ which is no longer one in $B' \cup G$ is precisely the union of a covering set of boys in $B' \cup G'$ and $\{b\}$,

so the number of covering sets of boys in $B \cup G$ is the sum of those in $B' \cup G$ and $B' \cup G'$.

On the other hand,

(1′) a covering set of girls in $B \cup G$ is still one in $B' \cup G$; and

(2′) a covering set of girls in $B' \cup G$ which is no longer one in $B \cup G$ is precisely a covering set of girls in $B' \cup G'$,

so the number of covering sets of girls in $B \cup G$ is the difference of those in $B' \cup G$ and $B' \cup G'$.

Since the assertion is true for both $B' \cup G$ and $B' \cup G'$ by the induction hypothesis, the conclusion follows.

Solution 3. (Géza Kós) Let B and G denote the sets of boys and girls, respectively. For every pair $(b,g) \in B \times G$, write $f(b,g) = 0$ if they know each other, and $f(b,g) = 1$ otherwise. A set X of boys is covering if and only if

$$\prod_{g\in G}\left(1-\prod_{b\in X} f(b,g)\right) = 1.$$

Hence the number of covering sets of boys is

$$\begin{aligned}\sum_{X\subseteq B}\prod_{g\in G}\left(1-\prod_{b\in X} f(b,g)\right) &\equiv \sum_{X\subseteq B}\prod_{g\in G}\left(1+\prod_{b\in X} f(b,g)\right)\\ &= \sum_{X\subseteq B}\sum_{Y\subseteq G}\prod_{b\in X}\prod_{g\in Y} f(b,g) \pmod 2.\end{aligned}$$

By symmetry, the same is valid for the number of covering sets of girls.

Problem 2. Given a triangle ABC, let D, E, and F respectively denote the midpoints of the sides BC, CA, and AB. The circle BCF and the line BE meet again at P, and the circle ABE and the line AD meet again at Q. Finally, the lines DP and FQ meet at R. Prove that the centroid G of the triangle ABC lies on the circle PQR.

Solution 1. We will use the following lemma.

Lemma. *Let AD be a median in triangle ABC. Then $\cot\angle BAD = 2\cot A + \cot B$ and $\cot\angle ADC = \frac{1}{2}(\cot B - \cot C)$.*

Proof. Let CC_1 and DD_1 be the perpendiculars from C and D to AB. Using the signed lengths we write

$$\cot BAD = \frac{AD_1}{DD_1} = \frac{(AC_1 + AB)/2}{CC_1/2} = \frac{CC_1 \cot A + CC_1(\cot A + \cot B)}{CC_1} = 2\cot A + \cot B.$$

Similarly, denoting by A_1 the projection of A onto BC, we get

$$\cot ADC = \frac{DA_1}{AA_1} = \frac{BC/2 - A_1C}{AA_1} = \frac{(AA_1 \cot B + AA_1 \cot C)/2 - AA_1 \cot C}{AA_1} = \frac{\cot B - \cot C}{2}.$$

The Lemma is proved.

Turning to the solution, by the Lemma we get

$$\begin{aligned}\cot\angle BPD &= 2\cot\angle BPC + \cot\angle PBC = 2\cot\angle BFC + \cot\angle PBC \quad \text{(from circle } BFPC\text{)}\\ &= 2\cdot\frac{1}{2}(\cot A - \cot B) + 2\cot B + \cot C = \cot A + \cot B + \cot C.\end{aligned}$$

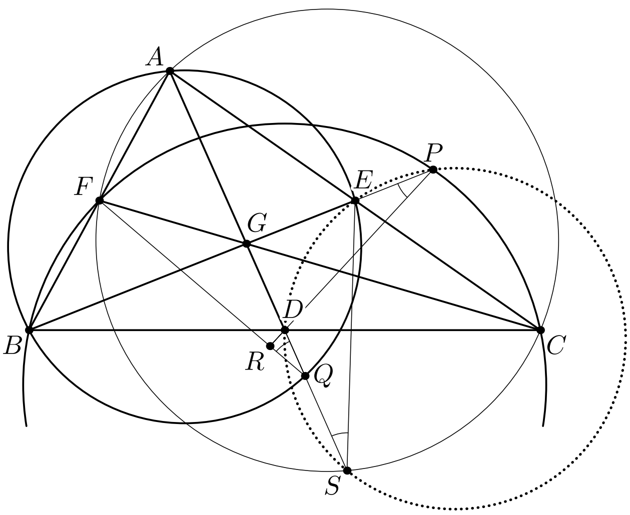
Similarly, $\cot\angle GQF = \cot A + \cot B + \cot C$, so $\angle GPR = \angle GQF$ and $GPRQ$ is cyclic.

Remark. The angle $\angle GPR = \angle GQF$ is the Brocard angle.

Solution 2. (Ilya Bogdanov and Marian Andronache) We also prove that $\angle(RP, PG) = \angle(RQ, QG)$, or $\angle(DP, PG) = \angle(FQ, QG)$.

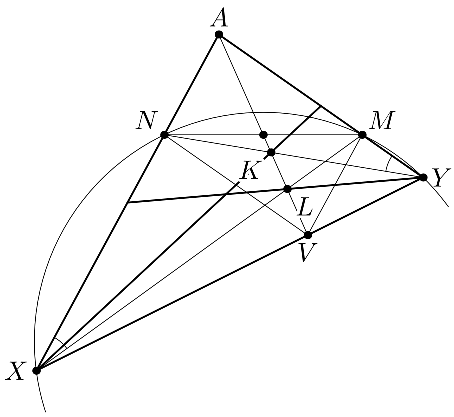
Let S be the point on ray GD such that $AG \cdot GS = CG \cdot GF$ (so the points A, S, C, F are concyclic). Then $GP \cdot GE = GP \cdot \frac{1}{2}GB = \frac{1}{2}CG \cdot GF = \frac{1}{2}AG \cdot GS = GD \cdot GS$, hence the points E, P, D, S are also concyclic, and $\angle(DP, PG) = \angle(GS, SE)$. The problem may therefore be rephrased as follows:

Given a triangle ABC, let D, E and F respectively denote the midpoints of the sides BC, CA and AB. The circle ABE, respectively, ACF, and the line AD meet again at Q, respectively, S. Prove that $\angle AQF = \angle ASE$ (and $ES = FQ$).



Upon inversion of pole A, the problem reads:

Given a triangle $AE'F'$, let the symmedian from A meet the medians from E' and F' at $K = Q'$ and $L = S'$, respectively. Prove that the angles $AE'L$ and $AF'K$ are congruent.

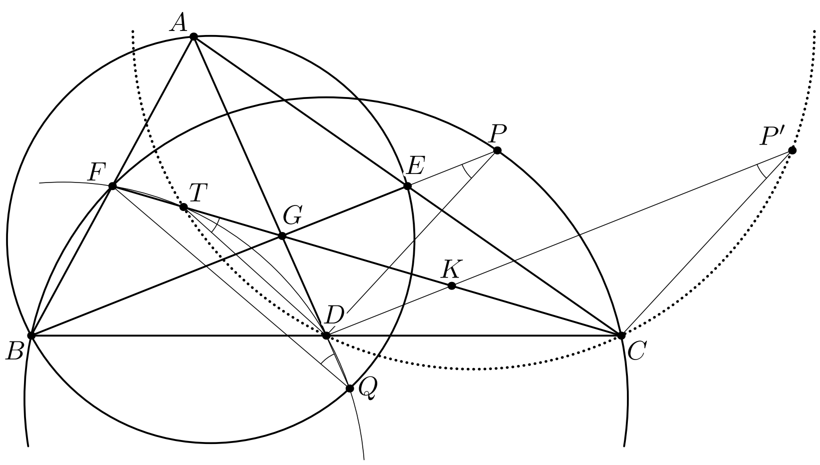


To prove this, denote $E' = X$, $F' = Y$. Let the symmedian from A meet the side XY at V and let the lines XL and YK meet the sides AY and AX at M and N, respectively. Since the points K and L lie on the medians, we have $VM \parallel AX$, $VN \parallel AY$. Hence $AMVN$ is a parallelogram, the symmedian AV of triangle AXY supports the median of triangle AMN, which implies that the triangles AMN and AXY are similar. Hence the points M, N, X, Y are concyclic, and $\angle AXM = \angle AYN$, QED.

Remark 1. We know that the points X, Y, M, N are concyclic. Invert back from A and consider the circles AFQ and AES: the former meets AC again at M' and the latter meets AB again at N'. Then the points E, F, M', N' are concyclic.

Remark 2. The inversion at pole A also allows one to show that $\angle AQF$ is the Brocard angle, thus providing one more solution. In our notation, it is equivalent to the fact that the points Y, K, and Z are collinear, where Z is the Brocard point (so $\angle ZAX = \angle ZYA = \angle ZXY$). This is valid because the lines AV, XK, and YZ are the radical axes of the following circles: (i) passing through X and tangent to AY at A; (ii) passing through Y and tangent to AX at A; and (iii) passing through X and tangent to AY at Y. The point K is the radical center of these three circles.

Solution 3. (Ilya Bogdanov) Again, we will prove that $\angle(DP, PG) = \angle(FQ, QG)$. Mark a point T on the ray GF such that $GF \cdot GT = GQ \cdot GD$; then the points F, Q, D, T are concyclic, and $\angle(FQ, QG) = \angle(TG, TD) = \angle(TC, TD)$.



Shift the point P by the vector $\overrightarrow{BD}$ to obtain point P'. Then $\angle(DP, PG) = \angle(CP', P'D)$, and we need to prove that $\angle(CP', P'D) = \angle(CT, TD)$. This is precisely the condition that the points T, D, C, P' be concyclic.

Denote $GE = x$, $GF = y$. Then $GP \cdot GB = GC \cdot GF$, so $GP = y^2/x$. On the other hand, $GB \cdot GE = GQ \cdot GA = 2GQ \cdot GD = 2GT \cdot GF$, so $GT = x^2/y$. Denote by K the point of intersection of DP' and CT; we need to prove that $TK \cdot KC = DK \cdot KP'$.

Now, $DP' = BP = BG+GP = 2x+y^2/x$, $CT = CG+GT = 2y+x^2/y$, $DK = BG/2 = x$, $CK = CG/2 = y$. Hence the desired equality reads $x(x + y^2/x) = y(y + x^2/y)$ which is obvious.

Remark. The points B, T, E, and C are concyclic, hence the point T is also of the same kind as P and Q.

Problem 3. Each positive integer number is coloured red or blue. A function f from the set of positive integer numbers into itself has the following two properties:

(a) if $x \leq y$, then $f(x) \leq f(y)$; and

(b) if x, y and z are all (not necessarily distinct) positive integer numbers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integer numbers x.

Solution. For integer x, y, by a segment $[x, y]$ we always mean the set of all integers t such that $x \leq t \leq y$; the *length* of this segment is $y - x$.

If for every two positive integers x, y sharing the same colour we have $f(x)/x = f(y)/y$, then one can choose $a = \max\{f(r)/r, f(b)/b\}$, where r and b are arbitrary red and blue numbers, respectively. So we can assume that there are two red numbers x, y such that $f(x)/x \neq f(y)/y$.

Set $m = xy$. Then each segment of length m contains a blue number. Indeed, assume that all the numbers on the segment $[k, k + m]$ are red. Then

$$f(k + m) = f(k + xy) = f(k + x(y - 1)) + f(x) = \cdots = f(k) + yf(x),$$
$$f(k + m) = f(k + xy) = f(k + (x - 1)y) + f(y) = \cdots = f(k) + xf(y),$$

so $yf(x) = xf(y)$ — a contradiction. Now we consider two cases.

Case 1. Assume that there exists a segment $[k, k + m]$ of length m consisting of blue numbers. Define $D = \max\{f(k), \ldots, f(k + m)\}$. We claim that $f(z) - f(z - 1) \leq D$, whatever $z > k$, and the conclusion follows. Consider the largest blue number b_1 not exceeding z, so $z - b_1 \leq m$, and some blue number b_2 on the segment $[b_1 + k, b_1 + k + m]$, so $b_2 > z$. Write $f(b_2) = f(b_1) + f(b_2 - b_1) \leq f(b_1) + D$ to deduce that $f(z + 1) - f(z) \leq f(b_2) - f(b_1) \leq D$, as claimed.

Case 2. Each segment of length m contains numbers of both colours. Fix any red number $R \geq 2m$ such that $R + 1$ is blue and set $D = \max\{f(R), f(R + 1)\}$. Now we claim that $f(z + 1) - f(z) \leq D$, whatever $z > 2m$. Consider the largest red number r not exceeding z and the largest blue number b smaller than r; then $0 < z - b = (z - r) + (r - b) \leq 2m$, and $b + 1$ is red. Let $t = b + R + 1$; then $t > z$. If t is blue, then $f(t) = f(b) + f(R + 1) \leq f(b) + D$, and $f(z + 1) - f(z) \leq f(t) - f(b) \leq D$. Otherwise, $f(t) = f(b + 1) + f(R) \leq f(b + 1) + D$, hence $f(z + 1) - f(z) \leq f(t) - f(b + 1) \leq D$, as claimed.