ALGEBRA

A1. Let m and n be integers greater than 2, and let A and B be non-constant polynomials with complex coefficients, at least one of which has a degree greater than 1. Prove that, if the degree of the polynomial $A^m - B^n$ is less than $\min(m, n)$, then $A^m = B^n$.

Indonesia, Tobi Moektijono

Solution. The conclusion is a consequence of the following version of the Mason–Stothers theorem:

Theorem. *If f and g are coprime polynomials with complex coefficients, and $\deg f + \deg g > 0$, then $\max(\deg f, \deg g)$ is less than the number of distinct roots of the polynomial $fg(f+g)$.*

Back to the problem, we first show that A and B are not coprime. Otherwise, let $f = A^m$ and $g = -B^n$ in the Mason–Stothers theorem to infer that $m \deg A$ and $n \deg B$ are both at most $\deg A + \deg B + \deg(A^m - B^n) - 1$, so

$$\begin{aligned}\deg(A^m - B^n) &\geq (m/2 - 1)\deg A + (n/2 - 1)\deg B + 1 \\ &\geq (m/2 - 1) + (n/2 - 1) + (\min(m, n)/2 - 1) + 1 \\ &> (m + n)/2 - 1 \geq \min(m, n) - 1,\end{aligned}$$

contradicting the condition in the statement.

Consequently, $C = \gcd(A, B)$ is not constant. Since $C^{\min(m,n)}$ divides $A^m - B^n$, and

$$\deg C^{\min(m,n)} \geq \min(m, n) > \deg(A^m - B^n),$$

it follows that $A^m = B^n$.

Remark. The conclusion may fail if $\deg A = \deg B = 1$. For instance, $\deg\big((X+1)^k - X^k\big) = k - 1 < k = \min(k, k)$. The conclusion still holds in this case, provided that $\deg(A^m - B^n) < \min(m, n) - 1$. The argument in the solution applies verbatim.

COMBINATORICS

C1. Call a point in the Cartesian plane with integer coordinates a *lattice point*. Given a finite set $\mathcal{S}$ of lattice points we repeatedly perform the following operation: given two distinct lattice points A, B in $\mathcal{S}$ and two distinct lattice points C, D not in $\mathcal{S}$ such that $ACBD$ is a parallelogram with $AB > CD$, we replace A, B by C, D. Show that only finitely many such operations can be performed.

United Kingdom, Joe Benton

Solution. We fix a lattice point O and show that the integer $\sum_{X\in\mathcal{S}} OX^2$ decreases each time an operation is performed, so the process must stop.

Indeed, if A, B are replaced by C, D, then

$$\begin{aligned} OA^2 + OB^2 &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})^2 + \frac{1}{2}(\overrightarrow{OA} - \overrightarrow{OB})^2 = \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD})^2 + \frac{1}{2}AB^2 \\ &> \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD})^2 + \frac{1}{2}CD^2 = \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD})^2 + \frac{1}{2}(\overrightarrow{OC} - \overrightarrow{OD})^2 = OC^2 + OD^2, \end{aligned}$$

while the other summands are left unchanged. This ends the proof.

Remarks. (1) One may notice that the computation in the solution above is just a double application of Apollonius' theorem which expresses the length of a median in a triangle via its sidelengths.

(2) An alternative integer monovariant is $\sum_{X,Y\in\mathcal{S}} XY^2$. To check this, notice that the only terms that change when we replace A, B with C, D are AB^2 (replaced with CD^2, which must be smaller), and those of the form $PA^2 + PB^2$ with $P \in \mathcal{S} \smallsetminus \{A, B\}$ (replaced by $PC^2 + PD^2$, which is smaller due to the same application of Apollonius' theorem).

C2. Fix integers $n \geq k \geq 2$. We call a collection of integral valued coins *n-diverse* if no value occurs in it more than n times. Given such a collection, a number S is *n-reachable* if that collection contains n coins whose sum of values equals S. Find the least positive integer D such that for any n-diverse collection of D coins there are at least k numbers that are n-reachable.

Bulgaria, Alexandar Ivanov

Solution. The required number is $D = n + k - 1$. We first show that $n + k - 2$ coins are not enough. Indeed, if a collection consists of n coins of value 1 and $k - 2 (< n)$ coins of value 2, then all n-reachable numbers have the form $x + 2y = n + y$ (since $x + y = n$), where $y \in \{0, 1, \ldots, k-2\}$. Thus, there are only $k - 1$ such numbers.

To prove that the number $D = n + k - 1$ suffices, we make use of the following lemma. In the sequel, we denote a coin and its value by the same letter.

Lemma. In any n-diverse collection of D coins one may find $D - n$ disjoint pairs, each consisting of two coins of distinct values.

Proof. Enumerate the coins $c_1 \leq c_2 \leq \cdots \leq c_D$ in a non-decreasing order according to their values. Then the desired pairs are (c_1, c_{n+1}), (c_2, c_{n+2}), $\ldots$, (c_{D-n}, c_D): they are clearly disjoint, and the equality $c_i = c_{n+i}$ would yield $c_i = c_{i+1} = \cdots = c_{i+n}$ which contradicts n-diversity of the collection. $\square$

Now, given any n-diverse collection of $D = n + k - 1$ coins, apply the lemma to find pairs (a_1, b_1), (a_2, b_2), $\ldots$, (a_{k-1}, b_{k-1}); the coins in each pair are listed so that $a_i < b_i$ for every $i = 1, 2, \ldots, k - 1$. Let $c_1, \ldots, c_{n-k+1}$ be the other coins in the collection.

Set $S = c_1 + c_2 + \cdots + c_{n-k+1}$. Then each of the k sums

$$S_m = S + \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{k-1} b_i, \qquad m = 0, 1, \ldots, k - 1,$$

is clearly n-reachable. Moreover, the inequalities imposed on the pairs yield $S_0 < S_1 < \cdots < S_{k-1}$, so we have found k distinct n-reachable numbers, as required.

Remark. One may notice that the ideas in the solution above are close to those used in the classical proof of the celebrated Erdős–Ginzburg–Ziv theorem.

C3. N teams take part in a league. Every team plays every other team exactly once during the league, and receives 2 points for each win, 1 point for each draw, and 0 points for each loss. At the end of the league, the sequence of total points in descending order $\mathcal{A} = (a_1 \geq a_2 \geq \cdots \geq a_N)$ is known, as well as which team obtained which score. Find the number of sequences $\mathcal{A}$ such that the outcome of all matches is uniquely determined by this information.

United Kingdom, Dominic Yeo

Solution. The required number is the $(N+1)$st Fibonacci number F_{N+1}, where $F_1 = F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$, $k \geq 3$; explicitly, $F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$.

In the sequel, a *tournament* is a collection of all game results. We say that a tournament is *deterministic* if its score sequence determines it uniquely. We always assume that the teams in a tournament are enumerated $t_1, t_2, \ldots, t_N$ in a non-decreasing order of their scores $a_1 \geq a_2 \geq \cdots \geq a_n$.

The argument is in three stages. First, we describe some deterministic tournaments and their score sequences. Then, we show no other tournament is deterministic. Finally, we enumerate the obtained tournaments.

Step 1. Let σ be some collection of disjoint pairs of adjacent elements of $\{1, \ldots, N\}$. Now we define a tournament T^σ by letting t_i and t_{i+1} make a draw if $(i, i+1) \in \sigma$, and letting t_i win t_j, where $i < j$, otherwise. In other words, we take a transitive tournament, and then clone several teams in it, letting the team draw with its clone.

The corresponding score sequence A^σ is defined as $a_i^\sigma = a_{i+1}^\sigma = 2N - (2i+1)$ if $(i, i+1) \in \sigma$, and $a_j^\sigma = 2(N-j)$ for all other indices j. For instance, $T^\varnothing$ is just the usual transitive tournament.

To show that T^σ is deterministic, we induct on N. The base cases $N = 1, 2$ are obvious. Now, if $(1,2) \notin \sigma$, then $a_1^\sigma = 2N - 2$, and so t_1 must have won all their matches. Since these matches contribute no points to the remaining $N-1$ teams, their scores in their internal sub-tournament are uniquely reconstructed. Since this sub-tournament is deterministic by the inductive hypothesis, the whole tournament is also deterministic.

Similarly, if $(1,2) \in \sigma$, then $a_1^\sigma + a_2^\sigma = 4N - 6$, and so t_1 and t_2 must both have defeated all other teams, and drawn with each other since $a_1^\sigma = a_2^\sigma$. Again, the application of the inductive hypothesis finishes the inductive step.

So, any score sequence $\mathcal{A}^\sigma$ uniquely determines the tournament T^σ.

Step 2. Consider any deterministic tournament T. Our aim is to show that it is indeed of the form T^σ described above. We write $x \to y$ and $x \leftrightarrow y$ to denote $\{x \text{ beats } y\}$ and $\{x, y \text{ draw}\}$ respectively.

Lemma. In every ordered triple of teams (x, y, z), one of them won their cyclic successor.

Proof. Assume the contrary. Then one may change the results of the three mentioned games letting each team get one more point in the game with their successor. This does not affect the score sequence, so the tournament was not deterministic. $\square$

The Lemma yields the following properties of a deterministic tournament.

(i) $a_i > a_j \iff t_i \to t_j$. Indeed, otherwise there would exist either a pair (i, j) with $t_j \to t_i$ yet $a_i \geq a_j$, or a pair (i, j) with $t_i \leftrightarrow t_j$ yet $a_i > a_j$. In both cases, there is a team t_k whose result in the game with t_j is strictly better than that with t_i. Therefore, the triple (t_i, t_j, t_k) violates the Lemma.

(ii) There are no three indices i, j, k with $a_i = a_j = a_k$. Otherwise, by *(i)* we get $t_i \leftrightarrow t_j \leftrightarrow t_k \leftrightarrow t_i$, so (t_i, t_j, t_k) violates the Lemma.

Now, given a deterministic tournament T, we denote by σ the set of all pairs (i,j) with $a_i = a_j$. By property (ii), these pairs are disjoint, and, by monotonicity, each consists of consecutive indices. Therefore, property (i) yields $T = T^\sigma$, as desired.

Step 3: Let $\mathcal{B}_N$ be the set of suitable σs corresponding to N. It remains just to count $B_N = |\mathcal{B}_N|$. There are natural bijections between $\{\sigma \in \mathcal{B}_N : (1,2) \notin \sigma\}$ and $\mathcal{B}_{N-1}$, and between $\{\sigma \in \mathcal{B}_N : (1,2) \in \sigma\}$ and $\mathcal{B}_{N-2}$.

Thus $B_N = B_{N-1} + B_{N-2}$. Now, it is clear that $B_1 = 1$, and after a bit of thought $B_2 = 2$. So $B_N = F_{N+1}$, the $(N+1)$st Fibonacci number.

Remark. One may perform Steps 2 and 3 in many different ways. Although the general scheme preserves, the arguments may vary. Below we list several such arguments.

Alternative approaches to Step 2. (a) The Lemma from the Solution above may be split into cases. E.g., one may list explicitly sub-tournaments on three teams that never appear in a deterministic tournament, namely: **(1)** $x \to y$, $y \to z$, $z \to x$; **(2)** $x \leftrightarrow y$, $y \leftrightarrow z$, $z \leftrightarrow x$; **(3)** $x \leftrightarrow y$, $y \leftrightarrow z$, $z \to x$; and **(4)** $y \to x$, $z \to y$, $x \leftrightarrow z$. (Indeed, the sub-tournaments **(1)** and **(2)** provide the same input to the score sequence, and so do **(3)** and **(4)**.) After that, a similar analysis may be performed in order to show that the only tournaments having no sub-tournaments of the forms **(1)**–**(4)** are those described in Step 1.

(b) An alternative combinatorial description of a deterministic tournament may also be obtained as follows. We realize a tournament as a directed (multi)graph on $[N] = \{1, 2, \dots, N\}$, where each pair of vertices are connected by two directed edges. These are both directed from loser to winner, else for a draw, we assign one edge in each direction. In this realization, the score sequence lists the in-degrees of the vertices.

Now, if the digraph includes a simple cycle with length at least 3, then one may obtain an alternative tournament with the same score sequence by reverting all arrows in the cycle. Thus, the only cycles a graph of a deterministic tournament may have are of length 2 (and correspond to draws). Thus, one may see that no team draws with two others (as they would form a cycle of length 3), and every pair which draws plays identically to each other team. So, after "de-cloning" such pairs, we get a usual transitive tournament, which establishes Step 2.

Such framework may also help in performing Step 1.

Alternative approach to Step 3. The number of relevant score sequences including exactly k pairs is $\binom{N-k}{k}$, for each $k = 0, 1, \dots, \lfloor N/2 \rfloor$. (Think about choosing k non-adjacent elements from $[N-1]$.)

Thus

$$\begin{aligned}
B_N &= \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} = \sum_{k=0}^{\lfloor N/2 \rfloor} \left[\binom{N-k-1}{k-1} + \binom{N-k-1}{k} \right] \\
&= \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{(N-2)-(k-1)}{k-1} + \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{(N-1)-k}{k} \\
&= \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{(N-2)-k}{k} + \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{(N-1)-k}{k} \\
&= B_{N-2} + B_{N-1},
\end{aligned}$$

and then one can finish as before.

It is worth mentioning that the obtained relation $\sum_k \binom{N-k}{k} = F_{N+1}$ is quite known, as well as the whole enumeration of partitions of $[N]$ into singletons and pairs of consecutive elements.

C4. Let k and s be positive integers such that $s < (2k+1)^2$. Initially, one cell out of an $n \times n$ grid is coloured green. On each turn, we pick some green cell c and colour green some s out of the $(2k+1)^2$ cells in the $(2k+1) \times (2k+1)$ square centred at c. No cell may be coloured green twice. We say that s is *k-sparse* if there exists some positive number C such that, for every positive integer n, the total number of green cells after any number of turns is always going to be at most Cn. Find, in terms of k, the least k-sparse integer s.

Nikolai Beluhov

Solution. The least sparse s is $s = 3k^2 + 2k$. We first show that $s = 3k^2 + 2k$ is sparse, then prove that no $s < 3k^2 + 2k$ is sparse. For a cell c, denote by $\mathcal{N}(c)$ its *neighbourhood*, i.e., the $(2k+1) \times (2k+1)$ square centred at c.

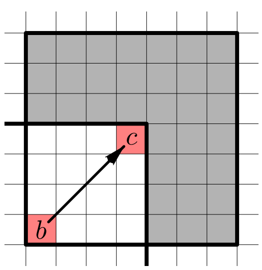
Part 1: $s = 3k^2 + 2k$ *is sparse.*

Recolour red the initial cell, and the centres of all turns during the game. Draw an arrow from every red cell c to all red cells that become green on c's turn. This results in an oriented tree G with red cells as vertices.

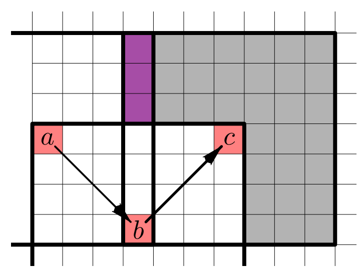
Say that a vertex a of G *precedes* another vertex b of G if there exists an oriented path in G from a to b. For every red cell c, let $S(c)$ be the number of cells in $\mathcal{N}(c)$ that are not in the neighbourhood of any cell that precedes c. Notice here that every coloured cell is accounted for at least one of the $S(c)$.

We have $S(\text{initial cell}) = (2k+1)^2$. Any other red cell c is the target of an arrow from some red cell b. The neighborhoods of b and c share at least $(k+1)^2$ cells, so $S(c) \leq (2k+1)^2 - (k+1)^2 = s$.

Let us examine the cases when this bound is achieved. Firstly, this may happen only if $\overrightarrow{bc} = (\pm k, \pm k)$; call such arrows *normal*, and all other arrows *short*. Moreover, if there is a *normal* in-arrow to b, this arrow should come from a cell a with $\overrightarrow{ab} = \overrightarrow{bc}$, otherwise $\mathcal{N}(a)$ and $\mathcal{N}(c)$ meet outside $\mathcal{N}(b)$. If the (normal) arrow $a \to b$ violates this last condition, we say that the arrow $b \to c$ is *jammed*.



A normal arrow



A jammed normal arrow

To summarize, a non-initial red cell c satisfies $S(c) = s$ only if its in-arrow is normal and not jammed. (Otherwise, as one can see, $S(c) \leq s - k$, but this improvement is superfluous for the solution.)

Let now v be the number of red cells, and let x be the total number of non-normal and jammed normal arrows. The total number of coloured cells is at least $vs + 1$; on the other hand, this number does not exceed

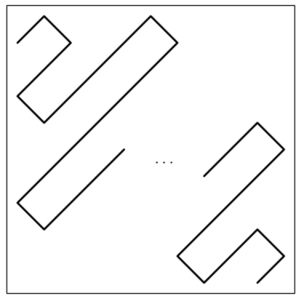
$$(2k+1)^2 + (v - x - 1)s + x(s-1) = vs + (k+1)^2 - x.$$

Comparing these two bounds, we obtain $x \leq (k+1)^2 - 1$.

Now remove all x exceptional arrows. After that, we are going to have at most $x + 1 \leq (k+1)^2$ (weakly) connected components, with all remaining arrows being normal and non-jammed. This means that the cells in each component are translates of each other by the same vector $(\pm k, \pm k)$, so every component contains at most $\lceil n/k \rceil \le n$ cells. Therefore, $v \le n(k+1)^2$, and hence the total number of green cells does not exceed $(2k+1)^2 v \le (k+1)^2(2k+1)^2 n$. Thus, s is sparse.

Part 2: *No $s < 3k^2 + 2k$ is sparse.*

To this end, set up graph G to be a path that zigzags all over the grid, whose arrows are all normal, and jammed arrows are always at least $2k$ steps apart — see the figure below. Even the red cells in this path have a density bounded away from 0, that is, there are at least Cn^2 of them for some positive constant C. So, it suffices to show that this graph G can be realized via the process in the problem, which is what we aim to do. For this purpose, we may — and will — assume that s takes its maximal value, i.e., $s = (2k+1)^2 - (k+1)^2 - 1$.



Zigzag path

Let $c_1 \to c_2 \to \cdots \to c_D$ be our path. We say that $\mathcal{P}(c_i) = \mathcal{N}(c_i) \setminus \mathcal{N}(c_{i+1})$ is the *preference set* for c_i (notice that $\mathcal{P}(c_i)$ contains $s+1$ cells). This means that, when performing a move centred at c_i, we colour c_{i+1}, and as many cells in $\mathcal{P}(c_i)$ as possible. In particular, if c_i belongs to the initial straight segment of the path, we colour just c_{i+1} and some cells outside $\mathcal{N}(c_{i+1})$.

Now let us check what happens after a jammed arrow $c_{j-1} \to c_j$ appears. The set $\mathcal{P}(c_j)$ meets $\mathcal{P}(c_{j-2})$ by k cells which could have been coloured before. Thus, we may need to colour at most $k-1$ cells (except for c_{j+1}) in $\mathcal{N}(c_{j+1})$ on the jth move. Similarly, on the $(j+1)$st move we colour at most $k-2$ cells in $\mathcal{N}(c_{j+2})$ (except for c_{j+2} itself), and so on. The figure below exemplifies this process for $k=3$.

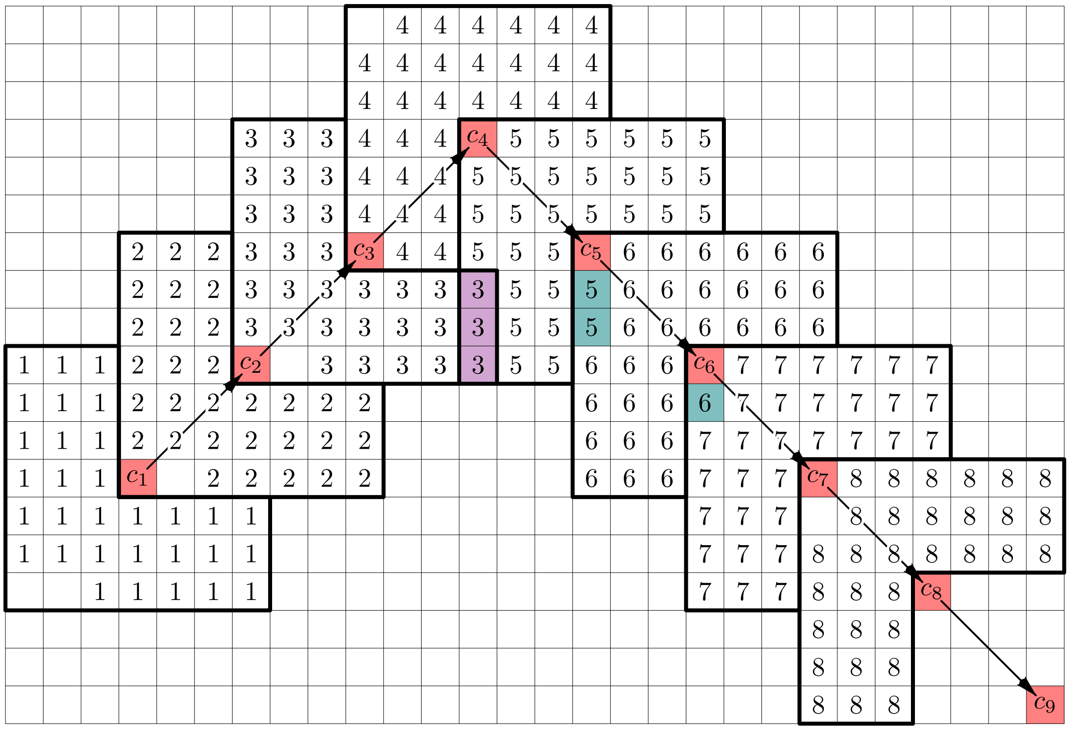
Since the jammed arrows are at least $2k$ steps apart, if $c_\ell \to c_{\ell+1}$ is the next jammed arrow, we colour just $c_{\ell+1}$ and some cells in $\mathcal{P}(c_\ell)$ on the ℓth turn. Therefore, we may proceed on in the way described above. This completes the proof.

Remark. We provide a sketch of a different (though ideologically similar) proof of Part 1, visualizing the counting argument.

Let us modify the process in the following way. When we pick a green cell c and color green s cells in its neighbourhood, we also recolour c red (as above), and we colour yellow all other yet uncoloured cells in this neighbourhood. This way, a yellow cell may be recoloured green, but a cell cannot be coloured green twice, as before. We also use the construction of graph G from the solution.

At any moment in the process, we denote by Y the total number of yellow cells. After the first turn, we have $s+1$ green, so $Y = (2k+1)^2 - (s+1) = (k+1)^2 - 1$. On each next turn centred at some cell c, the square $\mathcal{N}(c)$ already contains at least $(k+1)^2$ coloured cells, so the total number of coloured cells increases by at most s. Since the number of green cells increases by exactly s, this yields that the value of Y does not increase.

Since the initial value of Y does not depend on n, we need many turns on which Y remains constant. On such turn centred at c, the previously coloured cells in $\mathcal{N}(c)$ must form only a $(k+1) \times (k+1)$ square at a corner of $\mathcal{N}(c)$. As the above analysis shows, this implies that the in-arrow at c must be normal and non-jammed.



Exemplified process

Thus, the total number x of non-normal and jammed normal turns does not exceed the initial value of Y, i.e., $(k+1)^2-1$. Now one may finish as in the solution above.

The concept of yellow cells may also help in constructing an example for Part 2. Say, in the case $s=(2k+1)^2-(k+1)^2-1$, a turn may *increase* the value of Y by 1, if the previously coloured cells in $\mathcal{N}(c)$ form exactly a $(k+1)\times(k+1)$ corner (call such a turn *perfect*). When a zigzag path rotates, the value of Y decreases by k; so we just need k perfect turns in order to recover the value. This is what happens in a sample process described in the solution.

GEOMETRY

G1. Let ABC be a triangle and let H be the orthogonal projection of A on the line BC. Let K be the point on the segment AH such that $AH = 3KH$. Let O be the circumcentre of the triangle ABC and let M and N be the midpoints of the sides AC and AB, respectively. The lines KO and MN meet at a point Z and the perpendicular at Z to OK meets the lines AC and AB at X and Y, respectively. Show that $\angle XKY = \angle CKB$.

ITALY

Solution. Let L be the midpoint of BC, and let D be the projection of L onto MN; the points L, O, and D are collinear.

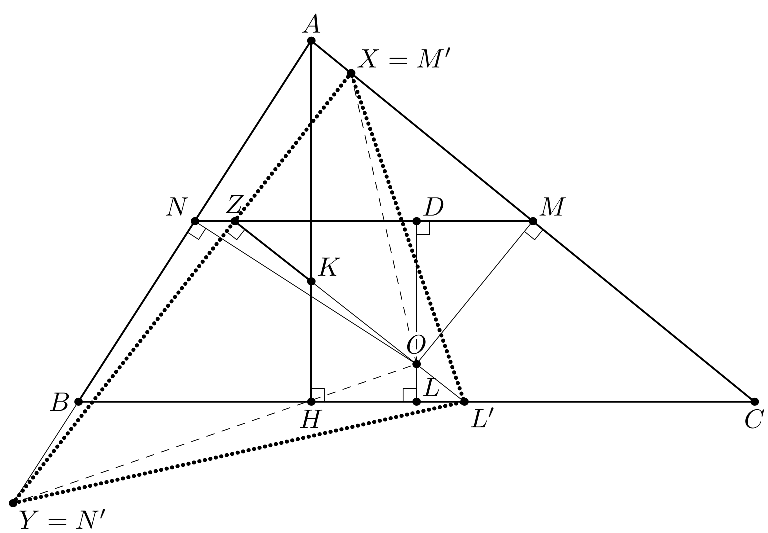
Consider the rotational homothety centred at O and mapping D to Z. Images under this transform will be denoted by primes; so, e.g., $Z = D'$. For every point P, the triangle OPP' is similar to ODD'; so, in particular, $\angle OPP' = 90°$.

Since $\angle ODM = 90°$, we have $\angle OZM' = 90°$; since $\angle OMM' = 90°$, the point M' lies on AC. So $M' = XY \cap AC = X$. Similarly, $N' = Y$.

Since $\angle OLL' = 90°$, the point L' lies on BC. Since $O \in LD$, we get $O \in L'Z$, so the points L', Z, O, and K are all collinear. Let E be the midpoint of AH. Since $MN \parallel BC$, we get

$$\frac{L'K}{KZ} = \frac{HK}{KE} = 2.$$

Finally, notice that the triangles $L'XY$ and ABC are similar (since they are both similar to LMN). The point K splits their altitudes $L'Z$ and AH in the same ratio, so K corresponds to itself in these triangles. Therefore, the triangles BKC and XKY are also similar, which yields the required equality.



Remarks. There exist many variations of the solution above. Here we list some of them.

Firstly, the fact that the triangles OXY and OMN are similar follows from angle chasing in cyclic quadrilaterals $OMXZ$ and $ONYZ$. This suggests using the rotational homothety defined above, but one may very well proceed without it.

One useful consequence of the mentioned similarity is that the points A, X, Y, and O are concyclic (this also follows from the converse of Simson's theorem applied to the point O in the triangle AXY).

Now, the point L' may alternatively be defined as the point where the circles OCX and OBY meet again (by Miquel's theorem, this point lies on BC). By angle chasing, one may obtain $L' \in OZ$; another angle chasing shows that the triangles $L'XY$ and ABC are similar. Then one may finish as in the solution above.

G2. Let ABC be a triangle, let S and T be the midpoints of the sides BC and CA, respectively, and let M be the midpoint of the segment ST. The circle ω through A, M and T meets the line AB again at N, and the tangents of ω at M and N meet at P. Prove that P lies on the line BC if and only if the triangle ABC is isosceles with apex at A.

Indonesia, Reza Kumara

Solution 1. Use complex coordinates; as usual, a lower case Roman letter denotes the complex coordinate of the point denoted by the corresponding upper case Roman letter.

Let ω be the standard unit circle centred at the origin O and let $t = 1$, so $c = 2 - a$ and $s = 2m - 1$. Since AN and MT are parallel, we have $\angle AOT = \angle MON$ and hence $a/1 = m/n$, or $n = m\overline{a}$. Plug this into $p = 2mn/(m + n)$ to get $p = 2m/(a + 1)$.

Now, P lies on the line BC if and only if $(p - s)/(s - c)$ is real. Using the above expressions for s and p, and carrying out calculations, this is the case if and only if

$$\frac{-2am + a + 1}{(a + 1)(2m + a - 3)}$$

is real; that is, it is equal to its complex conjugate. Carrying out calculations, this is equivalent to $(a - 1)(a + \overline{a} + 6) = 4(a - 1)(m + \overline{m})$, i.e., $a + \overline{a} + 6 = 4(m + \overline{m})$, since $A \neq T$, so $a \neq t = 1$. Alternatively, but equivalently, $(a - 1)(\overline{a} - 1) = 4(m - 1)(\overline{m} - 1)$, or $|a - 1| = 2|m - 1|$, which is the case if and only if $AT = 2MT$. The latter is clearly equivalent to $AC = AB$. This ends the proof.

Remark. The equivalence still holds if P is an ideal point. (It is not hard to see that the necessary and sufficient geometric condition for P to be an ideal point is that $AB + 2AC \cos A = 0$.) In this case, $n = -m$, $a = -1$, $c = 3$ and $b = 2s - c = 2(2m - 1) - 3 = 4m - 5$. For P to be the ideal point of the line BC it is necessary and sufficient that MN be perpendicular to BC; that is, $(b - c)/(m - n) = 2(1 - 2\overline{m})$ be purely imaginary. Alternatively, but equivalently, $m + \overline{m} = 1$, i.e., $m^2 - m + 1 = 0$. For such an m, $|b - a| = 4|m - 1| = 4|m|^2 = 4 = |c - a|$ and $|b - c| = 4|m - 2| = 4\sqrt{3}$, so the triangle ABC is isosceles with apex at A, and the internal angle at the apex is 120°.

Solution 2. To avoid case analysis, assume the angle BAC acute and N on the segment AB.

Notice that $ATMN$ is an isosceles trapezoid. Let CH be an altitude in the triangle ABC; then $\angle THA = \angle TAB = \angle MNA$, so $TH \parallel MN$, and $MNHT$ is a parallelogram. We

Let P' be the intersection of BC with the perpendicular bisector ℓ of MN (the case $MN \perp BC$, in which P' is not well-defined, can be treated easily). Since P also lies on ℓ, the conditions $P \in BC$ and $P = P'$ are equivalent. The latter condition may in turn be rewritten as $\angle PMN = \angle MAN$.

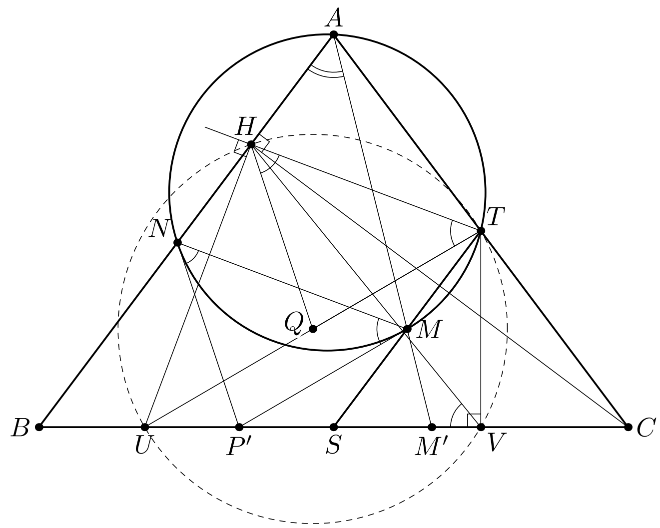
Choose U so that P is the midpoint of SU, and let Q be the midpoint of TU. Then the triangle THQ is obtained from MNP' by translation at $\overline{MT} = \overline{PQ}$, so $QP' = QT = QH$. Denoting by V the projection of T onto BC, we get $QT = QV$, so the points H, T, U, and V lie on a circle centred at Q. Therefore, $\angle PMN = \angle UTH = \angle UVH$. Thus, the condition $P = P'$ is equivalent to $\angle BVH = \angle MAN$, i.e., to the fact that the triangles BVH and BAM' are similar, where $M' = AM \cap BC$.

This last condition is in turn equivalent to

$$\frac{BM'}{BA} = \frac{BH}{BT}.$$

Using standard notation and noticing that $\dfrac{SM'}{M'C} = \dfrac{SM}{MT} \cdot \dfrac{TA}{AC} = \dfrac{1}{2}$ by Menelaus' theorem, we obtain

$$\frac{BM'}{BA} = \frac{2a/3}{c} \qquad \text{and} \qquad \frac{BH}{BV} = \frac{a \cos\beta}{(a + c\cos\beta)/2} = \frac{2a\cos\beta}{a + c\cos\beta},$$



so $P = P'$ if and only if

$$\frac{2a}{3c} = \frac{2a\cos\beta}{a + c\cos\beta} \iff 3c\cos\beta = a + c\cos\beta \iff a = 2c\cos\beta \iff b = c,$$

as required.

NUMBER THEORY

N1. Determine all polynomials f with integer coefficients such that $f(p)$ is a divisor of $2^p - 2$ for every odd prime p.

Italy

Solution 1. The required polynomials are ± 1, ± 2, ± 3, ± 6, $\pm X$, $\pm 2X$. It is readily checked that all these polynomials satisfy the condition in the statement.

Now let f be a polynomial satisfying the required condition, and assume first that $f(0) \neq 0$. Let S be the set of those primes p such that $p \mid f(p)$. Since $f(0) \neq 0$, the set S is finite, so $p \nmid f(p)$ for all but finitely many primes p. Let p be a large enough such prime, and suppose, if possible, that $f(p)$ is divisible by some prime $q > 3$. Refer to the Chinese remainder theorem to find an integer R with $R \equiv p \pmod q$ and $R \equiv -1 \pmod{q-1}$, and notice that R is coprime to $q(q-1)$. Now, by Dirichlet's theorem on primes in arithmetic sequences of integers, there exists a prime $r \neq p$ such that $r \equiv R \pmod{q(q-1)}$, that is, $r \equiv p \pmod q$ and $r \equiv -1 \pmod{q-1}$.

Clearly, $f(r) \equiv f(p) \equiv 0 \pmod q$, so $q \mid f(r) \mid 2(2^r - 2) = 2^{r+1} - 4$. Since $q > 2$, the condition $r + 1 \equiv 0 \pmod{q-1}$ and Fermat's little theorem imply $2^{r+1} \equiv 1 \pmod q$, hence $0 \equiv 2^{r+1} - 4 \equiv 1 - 4 \equiv -3 \pmod q$; that is, $q = 3$, contradicting the choice $q > 3$. Consequently, for every large enough prime p, the only primes that may divide $f(p)$ are 2 and 3.

On the other hand, $f(p) \mid 2^p - 2$, and $4 \nmid 2^p - 2$, so $v_2(f(p)) \leq 1$; moreover, if $p \equiv 5 \pmod 6$, then $2^p - 2 \equiv 2^5 - 2 \equiv 30 \equiv 3 \pmod 9$, so $v_3(f(p)) \leq 1$ for any such p. Consequently, $|f(p)| \leq 6$ for every large prime $p \equiv 5 \pmod 6$, so f is constant. Since this constant divides $2^3 - 2 = 6$, it is one of ± 1, ± 2, ± 3, ± 6.

Finally, we deal with the case $f(0) = 0$. Write $f = X^m \cdot g$ for some positive integer m and some polynomial g with integer coefficients such that $g(0) \neq 0$. Since g also satisfies the condition in the statement, it is one of ± 1, ± 2, ± 3, ± 6. The condition $f(3) \mid 2^3 - 2 = 6$ then forces $m = 1$ and $g = \pm 1$ or ± 2, so $f = \pm X$ or $\pm 2X$.

Solution 2. Only the case $f(0) \neq 0$ will be considered here; the case $f(0) = 0$ is dealt with as in Solution 1.

We show that there is an infinite set S of odd primes satisfying the following condition: For every p in S there exists an odd prime q such that $\gcd(p-1, q-1) = 2$ and $f(p)$ divides $f(q)$. For such a p, the value $f(p)$ divides $\gcd(2^p - 2, 2^q - 2) = 2\gcd(2^{p-1} - 1, 2^{q-1} - 1) = 2\left(2^{\gcd(p-1,q-1)} - 1\right) = 6$, so $f(p)$ is one of ± 1, ± 2, ± 3, ± 6. Since S is infinite, f is constant, so it is one of the eight numbers in the list above.

To prove existence of S, begin by noticing that $f(5)$ divides $30 = 2^5 - 2$, and 4 divides $f(5) - f(1)$, to infer that $f(1)$ is not divisible by 4, so $f(1) = 2m + 1$ or $f(1) = 2(2m + 1)$ for some integer m. Refer to the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic sequences of integers, to let S be the infinite set of primes $p > |f(0)|$, $p \equiv 3 \pmod 4$ and $p \equiv 2 \pmod{(2m+1)}$.

We now show that, for every p in S, there exists an odd prime q such that $\gcd(p-1, q-1) = 2$ and $f(p)$ divides $f(q)$. Consider a prime p in S. Since $(p-1)/2$ is odd and $p - 1$ and $2m + 1$ are relatively prime, $(p-1)/2$ is coprime to $f(1)$; and since $(p-1)/2$ divides $f(p) - f(1)$, it follows that $(p-1)/2$ is coprime to $f(p)$. Recall that $f(0) \neq 0$ and $p > |f(0)|$ to infer that $f(p)$ is also coprime to p, so there is an odd prime $q \equiv 2 \pmod{(p-1)/2}$ and $q \equiv p \pmod{f(p)}$, by the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic sequences of integers. Since $(p-1)/2$ is coprime to $q-1$, it follows that $\gcd(p-1, q-1) = 2\gcd((p-1)/2, (q-1)/2) = 2$; and since $f(p)$ divides $q - p$ which, in turn, divides $f(q) - f(p)$, it follows that $f(p)$ divides $f(q)$. This ends the proof.

N2. Prove that for each positive integer k there exists a number base b along with k triples of Fibonacci numbers (F_u, F_v, F_w) such that, when they are written in the base b, their concatenation $\overline{F_uF_vF_w}$ is also a Fibonacci number written in the base b. (Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n.)

Serbia

Solution. Recall that the sequence of *Lucas numbers* $L_1, L_2, \ldots$ is defined to satisfy the same Fibonacci recurrence relation $L_{n+2} = L_{n+1} + L_n$ starting with $L_0 = 2$, $L_1 = 1$. We extend the Fibonacci sequence in the negative direction by the same recurrence relation; notice that $F_{-k} = (-1)^{k-1}F_k$.

Recall (or prove by a straightforward induction) that $L_k = F_{k-1} + F_{k+1}$. We will make use of the fact that the *sparse* Fibonacci sequence $F_i, F_{i+s}, F_{i+2s}, \ldots$ satisfies the recurrence relation with Lucas numbers as coefficients, namely

$$F_{n+s} = L_sF_n - (-1)^sF_{n-s};$$

again, this relation may be proved by induction on s.

In view of the relation above, for every integers $k \geq 2$ and $i \geq 0$ we have

$$\begin{aligned} F_{4k+i} = L_{2k-1}F_{2k+i+1} + F_{i+2} &= L_{2k-1}(L_{2k-1}F_{i+2} + F_{i+3-2k}) + F_{i+2} \\ &= F_{i+2}L_{2k-1}^2 + (-1)^iF_{2k-i-3}L_{2k-1} + F_{i+2}. \end{aligned}$$

Thus, for every even $i = 0, 2, 4, \ldots, 2(k-1)$ we get that the L_{2k-1}-ary expansion of F_{4k+i} has exactly three digits, which are F_{i+2}, F_{2k-i-3}, and F_{i+2} (since all of them are nonnegative and do not exceed $F_{2k} < L_{2k-1}$). So, setting $b = L_{2k-1}$, we have shown the existence of k required triples having the form $(F_{i+2}, F_{2k-i-3}, F_{i+2})$.

Remarks. **(1)** The main difficulty of the problem seems to be in guessing a correct example. This may be done either by playing around small (although not very small) cases, or by playing with formulae involving sequences satisfying the Fibonacci relation. In both approaches, one does not need to be acquainted with Lucas numbers, although such acquaintance may help.

After the example has been guessed, one may proceed in different ways in order to prove it. E.g., the construction shown in the solution above may be established via explicit formulae $F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ and $L_n = \varphi^n + \psi^n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Namely, let n be odd and ℓ be even, with $\ell < n$ (in terms of the solution above, we have $n = 2k-1$ and $\ell = i+2$); then $(\varphi\psi)^n = -1$ and $(\varphi\psi)^\ell = 1$. Therefore, the base L_n number $\overline{F_\ell F_{n-\ell} F_\ell}$ equals

$$\begin{aligned} \overline{F_\ell F_{n-\ell} F_\ell} &= L_n^2F_\ell + L_nF_{n-\ell} + F_\ell = (L_n^2+1)F_\ell + L_nF_{n-\ell} \\ &= \left((\varphi^n + \psi^n)^2 + 1\right) \cdot \frac{\varphi^\ell - \psi^\ell}{\sqrt{5}} + (\varphi^n + \psi^n) \cdot \frac{\varphi^{n-\ell} - \psi^{n-\ell}}{\sqrt{5}} \\ &= \left(\varphi^{2n} - 1 + \psi^{2n}\right) \cdot \frac{\varphi^\ell - \psi^\ell}{\sqrt{5}} + (\varphi^n + \psi^n) \cdot \frac{\varphi^{n-\ell} - \psi^{n-\ell}}{\sqrt{5}} \\ &= \frac{\varphi^{2n+\ell} - \varphi^{2n-\ell} - \varphi^\ell + \psi^\ell + \psi^{2n-\ell} - \psi^{2n+\ell}}{\sqrt{5}} + \frac{\varphi^{2n-\ell} + \varphi^\ell - \psi^\ell - \psi^{2n-\ell}}{\sqrt{5}} \\ &= \frac{\varphi^{2n+\ell} - \psi^{2n+\ell}}{\sqrt{5}} = F_{2n+\ell}, \end{aligned}$$

as required.

(2) Some modifications of the problem are possible: since all the obtained triples consist of one-digit numbers, and the first and the last ones are equal, some of these conditions may be imposed in the problem statement. Such restrictions narrow the search, so they may make the problem easier.