The 5^{th} Romanian Master of Mathematics Competition

Solutions for the Day 2

Problem 4. Prove that there are infinitely many positive integer numbers n such that $2^{2^n+1}+1$ be divisible by n, but 2^n+1 be not.

Solution 1. Throughout the solution n stands for a positive integer. By Euler's theorem, $(2^{3^n}+1)(2^{3^n}-1) = 2^{2\cdot 3^n}-1 \equiv 0 \pmod{3^{n+1}}$. Since $2^{3^n}-1 \equiv 1 \pmod 3$, it follows that $2^{3^n}+1$ is divisible by 3^{n+1}.

The number $(2^{3^{n+1}}+1)/(2^{3^n}+1) = 2^{2\cdot 3^n} - 2^{3^n}+1$ is greater than 3 and congruent to 3 modulo 9, so it has a prime factor $p_n > 3$ that does not divide $2^{3^n}+1$ (otherwise, $2^{3^n} \equiv -1 \pmod{p_n}$, so $2^{2\cdot 3^n} - 2^{3^n}+1 \equiv 3 \pmod{p_n}$, contradicting the fact that p_n is a factor greater than 3 of $2^{2\cdot 3^n} - 2^{3^n}+1$).

We now show that $a_n = 3^n p_n$ satisfies the conditions in the statement. Since $2^{a_n}+1 \equiv 2^{3^n}+1 \not\equiv 0 \pmod{p_n}$, it follows that a_n does not divide $2^{a_n}+1$.

On the other hand, 3^{n+1} divides $2^{3^n}+1$ which in turn divides $2^{a_n}+1$, so $2^{3^{n+1}}+1$ divides $2^{2^{a_n}+1}+1$. Finally, both 3^n and p_n divide $2^{3^{n+1}}+1$, so a_n divides $2^{2^{a_n}+1}+1$.

As n runs through the positive integers, the a_n are clearly pairwise distinct and the conclusion follows.

Solution 2. (Géza Kós) We show that the numbers $a_n = (2^{3^n}+1)/9$, $n \geq 2$, satisfy the conditions in the statement. To this end, recall the following well-known facts:

(1) If N is an odd positive integer, then $\nu_3(2^N+1) = \nu_3(N)+1$, where $\nu_3(a)$ is the exponent of 3 in the decomposition of the integer a into prime factors; and

(2) If M and N are odd positive integers, then $(2^M+1, 2^N+1) = 2^{(M,N)}+1$, where (a,b) is the greatest common divisor of the integers a and b.

By (1), $a_n = 3^{n-1}m$, where m is an odd positive integer not divisible by 3, and by (2),

$$(m, 2^{a_n}+1) \mid (2^{3^n}+1, 2^{a_n}+1) = 2^{(3^n, a_n)}+1 = 2^{3^{n-1}}+1 < \frac{2^{3^n}+1}{3^{n+1}} = m,$$

so m cannot divide $2^{a_n}+1$.

On the other hand, $3^{n-1} \mid 2^{2^{a_n}+1}+1$, for $\nu_3(2^{2^{a_n}+1}+1) > \nu_3(2^{a_n}+1) > \nu_3(a_n) = n-1$, and $m \mid 2^{2^{a_n}+1}+1$, for $3^{n-1} \mid a_n$, so $3^n \mid 2^{a_n}+1$ whence $m \mid 2^{3^n}+1 \mid 2^{2^{a_n}+1}+1$. Since 3^{n-1} and m are coprime, the conclusion follows.

Remarks. There are several variations of these solutions. For instance, let $b_1 = 3$ and $b_{n+1} = 2^{b_n}+1$, $n \geq 1$, and notice that b_n divides b_{n+1}. It can be shown that there are infinitely many indices n such that some prime factor p_n of b_{n+1} does not divide b_n. One checks that for these n's the $a_n = p_n b_{n-1}$ satisfy the required conditions.

Finally, the numbers $3^n \cdot 571$, $n \geq 2$, form yet another infinite set of positive integers fulfilling the conditions in the statement — the details are omitted.

Solution 3. (Dušan Djukić) Assume that n satisfies the conditions of the problem. We claim that the number $N = 2^n+1 > n$ also satisfies these conditions.

Firstly, since $n \nmid N$, the fact (2) from Solution 2 allows to conclude that $2^n+1 \nmid 2^N+1$, or $N \nmid 2^N+1$. Next, since $n \mid 2^{2^n+1}+1 = 2^N+1$, we obtain from the same fact that $N = 2^n+1 \mid 2^{2^N+1}+1$, thus confirming our claim.

Hence, it suffices to provide only one example, hence obtaining an infinite series by the claim. For instance, one may easily check that the number $n = 57$ fits.

Problem 5. Given a positive integer number $n \geq 3$, colour each cell of an $n \times n$ square array one of $[(n+2)^2/3]$ colours, each colour being used at least once. Prove that the cells of some 1×3 or 3×1 rectangular subarray have pairwise distinct colours.

Solution. For more convenience, say that a subarray of the $n \times n$ square array *bears* a colour if at least two of its cells share that colour.

We shall prove that the number of 1×3 and 3×1 rectangular subarrays, which is $2n(n-2)$, exceeds the number of such subarrays, each of which bears some colour. The key ingredient is the estimate in the lemma below.

Lemma. *If a colour is used exactly p times, then the number of 1×3 and 3×1 rectangular subarrays bearing that colour does not exceed $3(p-1)$.*

Assume the lemma for the moment, let $N = [(n+2)^2/3]$ and let n_i be the number of cells coloured the ith colour, $i = 1, \ldots, N$, to deduce that the number of 1×3 and 3×1 rectangular subarrays, each of which bears some colour, is at most

$$\sum_{i=1}^{N} 3(n_i - 1) = 3\sum_{i=1}^{N} n_i - 3N = 3n^2 - 3N < 3n^2 - (n^2 + 4n) = 2n(n-2)$$

and thereby conclude the proof.

Back to the lemma, the assertion is clear if $p = 1$, so let $p > 1$.

We begin by showing that if a row contains exactly q cells coloured C, then the number r of 3×1 rectangular subarrays bearing C does not exceed $3q/2 - 1$; of course, a similar estimate holds for a column. To this end, notice first that the case $q = 1$ is trivial, so we assume that $q > 1$. Consider the incidence of a cell c coloured C and a 3×1 rectangular subarray R bearing C:

$$\langle c, R \rangle = \begin{cases} 1 & \text{if } c \subset R, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, given R, $\sum_c \langle c, R \rangle \geq 2$, and, given c, $\sum_R \langle c, R \rangle \leq 3$; moreover, if c is the leftmost or rightmost cell, then $\sum_R \langle c, R \rangle \leq 2$. Consequently,

$$2r \leq \sum_R \sum_c \langle c, R \rangle = \sum_c \sum_R \langle c, R \rangle \leq 2 + 3(q-2) + 2 = 3q - 2,$$

whence the conclusion.

Finally, let the p cells coloured C lie on k rows and ℓ columns and notice that $k + \ell \geq 3$, for $p > 1$. By the preceding, the total number of 3×1 rectangular subarrays bearing C does not exceed $3p/2 - k$, and the total number of 1×3 rectangular subarrays bearing C does not exceed $3p/2 - \ell$, so the total number of 1×3 and 3×1 rectangular subarrays bearing C does not exceed $(3p/2 - k) + (3p/2 - \ell) = 3p - (k + \ell) \leq 3p - 3 = 3(p-1)$. This completes the proof.

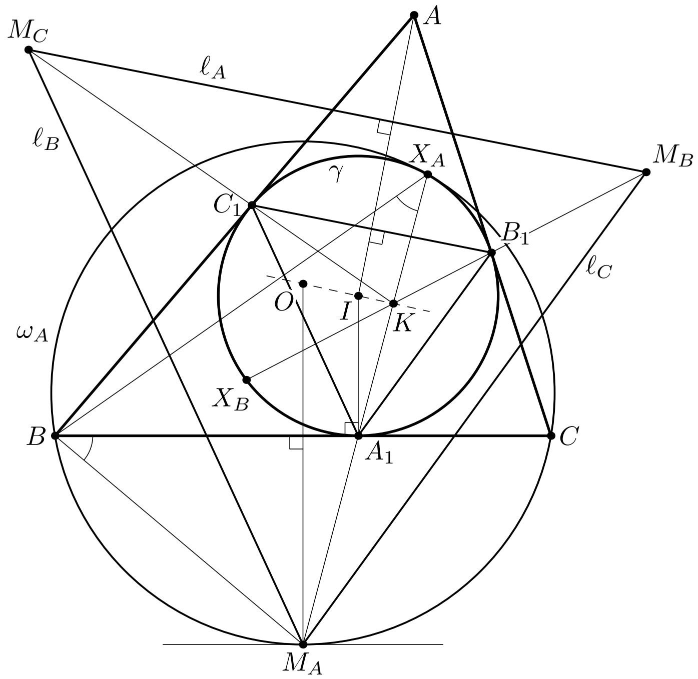
Remarks. In terms of the total number of cells, the number $N = [(n+2)^2/3]$ of colours is asymptotically close to the minimum number of colours required for some 1×3 or 3×1 rectangular subarray to have all cells of pairwise distinct colours, whatever the colouring. To see this, colour the cells with the coordinates (i, j), where $i + j \equiv 0 \pmod 3$ and $i, j \in \{0, 1, \ldots, n-1\}$, one colour each, and use one additional colour C to colour the remaining cells. Then each 1×3 and each 3×1 rectangular subarray has exactly two cells coloured C, and the number of colours is $\lceil n^2/3 \rceil + 1$ if $n \equiv 1$ or $2 \pmod 3$, and $\lceil n^2/3 \rceil$ if $n \equiv 0 \pmod 3$. Consequently, the minimum number of colours is $n^2/3 + O(n)$.

Problem 6. Let ABC be a triangle and let I and O respectively denote its incentre and circumcentre. Let ω_A be the circle through B and C and tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C through A meet again at A'; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Solution. Let γ be the incircle of the triangle ABC and let A_1, B_1, C_1 be its contact points with the sides BC, CA, AB, respectively. Let further X_A be the point of contact of the circles γ and ω_A. The latter circle is the image of the former under a homothety centred at X_A. This homothety sends A_1 to a point M_A on ω_A such that the tangent to ω_A at M_A is parallel to BC. Consequently, M_A is the midpoint of the arc BC of ω_A not containing X_A. It follows that the angles M_AX_AB and M_ABC are congruent, so the triangles M_ABA_1 and M_AX_AB are similar: $M_AB/M_AX_A = M_AA_1/M_AB$. Rewrite the latter $M_AB^2 = M_AA_1 \cdot M_AX_A$ to deduce that M_A lies on the radical axis ℓ_B of B and γ. Similarly, M_A lies on the radical axis ℓ_C of C and γ.

Define the points X_B, X_C, M_B, M_C and the line ℓ_A in a similar way and notice that the lines ℓ_A, ℓ_B, ℓ_C support the sides of the triangle $M_AM_BM_C$. The lines ℓ_A and B_1C_1 are both perpendicular to AI, so they are parallel. Similarly, the lines ℓ_B and ℓ_C are parallel to C_1A_1 and A_1B_1, respectively. Consequently, the triangle $M_AM_BM_C$ is the image of the triangle $A_1B_1C_1$ under a homothety Θ. Let K be the centre of Θ and let $k = M_AK/A_1K = M_BK/B_1K = M_CK/C_1K$ be the similitude ratio. Notice that the lines M_AA_1, M_BB_1 and M_CC_1 are concurrent at K.

Since the points A_1, B_1, X_A, X_B are concyclic, $A_1K \cdot KX_A = B_1K \cdot KX_B$. Multiply both sides by k to get $M_AK \cdot KX_A = M_BK \cdot KX_B$ and deduce thereby that K lies on the radical axis CC' of ω_A and ω_B. Similarly, both lines AA' and BB' pass through K.



Finally, consider the image O' of I under Θ. It lies on the line through M_A parallel to A_1I (and hence perpendicular to BC); since M_A is the midpoint of the arc BC, this line must be M_AO. Similarly, O' lies on the line M_BO, so $O' = O$. Consequently, the points I, K and O are collinear.

Remark 1. Many steps in this solution allow different reasonings. For instance, one may see that the lines A_1X_A and B_1X_B are concurrent at point K on the radical axis CC' of the circles ω_A and ω_B by applying Newton's theorem to the quadrilateral $X_AX_BA_1B_1$ (since the common tangents at X_A and X_B intersect on CC'). Then one can conclude that $KA_1/KB_1 = KM_A/KM_B$, thus obtaining that the triangles $M_AM_BM_C$ and $A_1B_1C_1$ are homothetical at K (and therefore K is the radical center of ω_A, ω_B, and ω_C). Finally, considering the inversion with the pole K and the power equal to $KX_1 \cdot KM_A$ followed by the reflection at P we see that the circles ω_A, ω_B, and ω_C are invariant under this transform; next, the image of γ is the circumcircle of $M_AM_BM_C$ and it is tangent to all the circles ω_A, ω_B, and ω_C, hence its center is O, and thus O, I, and K are collinear.

Remark 2. Here is an outline of an alternative approach to the first part of the solution. Let J_A be the excentre of the triangle ABC opposite A. The line J_AA_1 meets γ again at Y_A; let Z_A and N_A be the midpoints of the segments A_1Y_A and J_AA_1, respectively. Since the segment IJ_A is a diameter in the circle BCZ_A, it follows that $BA_1 \cdot CA_1 = Z_AA_1 \cdot J_AA_1$, so $BA_1 \cdot CA_1 = N_AA_1 \cdot Y_AA_1$. Consequently, the points B, C, N_A and Y_A lie on some circle ω'_A.

It is well known that N_A lies on the perpendicular bisector of the segment BC, so the tangents to ω'_A and γ at N_A and A_1 are parallel. It follows that the tangents to these circles at Y_A coincide, so ω'_A is in fact ω_A, whence $X_A = Y_A$ and $M_A = N_A$. It is also well known that the midpoint S_A of the segment IJ_A lies both on the circumcircle ABC and on the perpendicular bisector of BC. Since S_AM_A is a midline in the triangle A_1IJ_A, it follows that $S_AM_A = r/2$, where r is the radius of γ (the inradius of the triangle ABC). Consequently, each of the points M_A, M_B and M_C is at distance $R + r/2$ from O (here R is the circumradius). Now proceed as above.