The 6$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2. Grading schemes

Any equivalent approach, for each of the topics addressed in these Grading Schemes, should receive proper and proportionately judged equivalent marks.

Problem 4. Let P and P' be two convex quadrilateral regions in the plane (regions contain their boundary). Let them intersect, with O a point in the intersection. Suppose that for every line ℓ through O the segment $\ell \cap P$ is strictly longer than the segment $\ell \cap P'$. Is it possible that the ratio of the area of P' to the area of P is greater than 1.9?

Grading Scheme

A valid example of two desired polygons with a lack of proof that it works
.... **not less than 5 points**

A non-valid example which can easily be repaired .... **not more than 5 points**

If a valid example is not provided:

- Proving that $[P']/[P] \leq 2$ .... **1 point**
- Indicating **useful** properties of the desired configuration, e.g., that in order to satisfy the requirements, point O should be close enough to the boundary of P' .... **1 point**

Remark. These points are not to be added.

No points are awarded for just claiming the answer.

Problem 5. Given an integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest $x > a_{n-1}$ such that:

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor .$$

Prove that every prime occurs in the sequence $a_1, a_2, \ldots$.

GRADING SCHEME

Claiming that the a_n are exactly all kth-power-free numbers .... **2 points**

Claiming just that the a_n are among the kth-power-free numbers or vice versa .... **1 point**

Remark. These points are surely not no be added to each other, but they are to be added up to the points mentioned below.

Solution 1 is based on the relation

$$\sum_{b \in B,\, b \leq c} \left\lfloor \sqrt[k]{\frac{c}{b}} \right\rfloor = c, \tag{1}$$

where B is the set of all kth-power-free numbers.

- Proving that (1) holds .... **3 points**
- The final induction .... **2 points**

Solution 2 is based on the properties of the functions $f_n(x)$ and $g_n(x)$. It splits into two parts.

1. Preliminary observations:

- Just introducing the functions $f_n(x)$ and $g_n(x)$, and establishing that $f_n(x)+1 \geq f_n(x+1)$ .... **0 points**
- Proving that $f_n(x) < 0$ for all integer $x \in (a_{n-1}, a_n)$ .... **1 point**
- Proving that a_j/a_i is not the kth power of an integer .... **1 point**

2. The final induction. If, under the induction hypothesis, it is proved that

(1) a_n is the least number $x > a_{n-1}$ such that $g_n(x) = 0$ .... **1 point**

(2) the least number $x > a_{n-1}$ such that $g_n(x) = 0$ is kth-square-free .... **1 point**

Putting it all together .... **1 point**

Remark. The points for one approach are to be added up; the points for different approaches are not to be added up.

Problem 6. $2n$ distinct tokens are placed at the vertices of a regular $2n$-gon, with one token placed at each vertex. A *move* consists of choosing an edge of the $2n$-gon and interchanging the two tokens at the endpoints of that edge. Suppose that after a finite number of moves, every pair of tokens have been interchanged exactly once. Prove that some edge has never been chosen.

Grading Scheme

Showing that the circular order of any three tokens is reversed after the process has been carried out . . . . **1 point**

Showing that the final arrangement is obtained from the initial one by reflection in some line ℓ . . . . **1 point**

Proving that ℓ crosses two opposite sides a and b (but does not pass through the vertices) . . . . **1 point**

Proving that a token cannot move along both a and b . . . . **1 point**

Finally, showing that every two tokens either both moved along a or both moved along b . . . . **3 points**

If this is shown only for the tokens on the same side of ℓ . . . . **1 point is deducted**

All these points are to be added up.

No points are awarded for examples of such sequence of switchings.