The 6^{th} Romanian Master of Mathematics Competition

Solutions for the Day 2

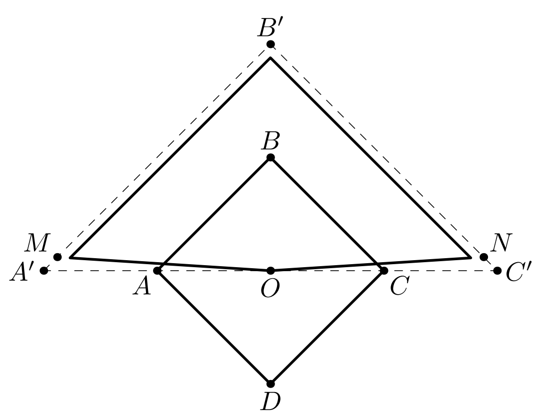
Problem 4. Let P and P' be two convex quadrilateral regions in the plane (regions contain their boundary). Let them intersect, with O a point in the intersection. Suppose that for every line ℓ through O the segment $\ell \cap P$ is strictly longer than the segment $\ell \cap P'$. Is it possible that the ratio of the area of P' to the area of P is greater than 1.9?

(Bulgaria) Nikolai Beluhov

Solution. The answer is in the affirmative: Given a positive $\epsilon < 2$, the ratio in question may indeed be greater than $2 - \epsilon$.

To show this, consider a square $ABCD$ centred at O, and let A', B', and C' be the reflections of O in A, B, and C, respectively. Notice that, if ℓ is a line through O, then the segments $\ell \cap ABCD$ and $\ell \cap A'B'C'$ have equal lengths, unless ℓ is the line AC.

Next, consider the points M and N on the segments $B'A'$ and $B'C'$, respectively, such that $B'M/B'A' = B'N/B'C' = (1-\epsilon/4)^{1/2}$. Finally, let P' be the image of the convex quadrangle $B'MON$ under the homothety of ratio $(1-\epsilon/4)^{1/4}$ centred at O. Clearly, the quadrangles $P \equiv ABCD$ and P' satisfy the conditions in the statement, and the ratio of the area of P' to the area of P is exactly $2 - \epsilon/2$.



Remarks. (1) With some care, one may also construct such example with a point O being interior for both P and P'. In our example, it is enough to replace vertex O of P' by a point on the segment OD close enough to O. The details are left to the reader.

(2) On the other hand, one may show that the ratio of areas of P' and P cannot exceed 2 (even if P and P' are arbitrary convex polygons rather than quadrilaterals). Further on, we denote by $[S]$ the area of S.

In order to see that $[P'] < 2[P]$, let us fix some ray r from O, and let r_α be the ray from O making an (oriented) angle α with r. Denote by X_α and Y_α the points of P and P', respectively, lying on r_α farthest from O, and denote by $f(\alpha)$ and $g(\alpha)$ the lengths of the segments OX_α and OY_α, respectively. Then

$$[P] = \frac{1}{2}\int_0^{2\pi} f^2(\alpha)\,d\alpha = \frac{1}{2}\int_0^{\pi} \left(f^2(\alpha) + f^2(\pi+\alpha)\right) d\alpha,$$

and similarly

$$[P'] = \frac{1}{2}\int_0^{\pi} \left(g^2(\alpha) + g^2(\pi+\alpha)\right) d\alpha.$$

But $X_\alpha X_{\pi+\alpha} > Y_\alpha Y_{\pi+\alpha}$ yields $2 \cdot \frac{1}{2}\left(f^2(\alpha)+f^2(\pi+\alpha)\right) = OX_\alpha^2 + OX_{\pi+\alpha}^2 \geq \frac{1}{2}X_\alpha X_{\pi+\alpha}^2 > \frac{1}{2}Y_\alpha Y_{\pi+\alpha}^2 \geq \frac{1}{2}(OY_\alpha^2 + OY_{\pi+\alpha}^2) = \frac{1}{2}\left(g^2(\alpha)+g^2(\pi+\alpha)\right)$. Integration then gives us $2[P] > [P']$, as needed.

This can also be proved via elementary methods. Actually, we will establish the following more general fact.

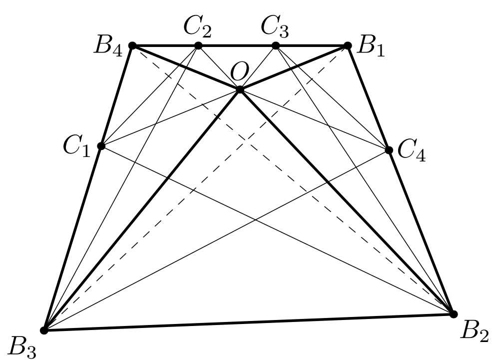
Fact. *Let $P = A_1A_2A_3A_4$ and $P' = B_1B_2B_3B_4$ be two convex quadrangles in the plane, and let O be one of their common points different from the vertices of P'. Denote by ℓ_i the line OB_i, and assume that for every $i = 1,2,3,4$ the length of segment $\ell_i \cap P$ is greater than the length of segment $\ell_i \cap P'$. Then $[P'] < 2[P]$.*

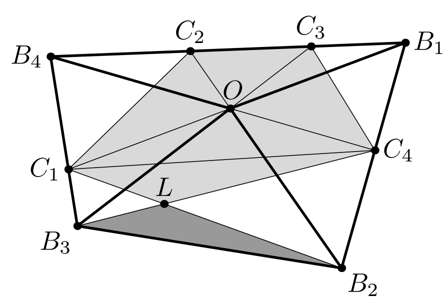
Proof. One of (possibly degenerate) quadrilaterals $OB_1B_2B_3$ and $OB_1B_4B_3$ is convex; the same holds for $OB_2B_3B_4$ and $OB_2B_1B_4$. Without loss of generality, we may (and will) assume that the quadrilaterals $OB_1B_2B_3$ and $OB_2B_3B_4$ are convex.

Denote by C_i such a point that $\ell_i \cap P'$ is the segment B_iC_i; let a_i be the length of $\ell_i \cap P$, and let α_i be the angle between ℓ_i and ℓ_{i+1} (hereafter, we use the cyclic notation, thus $\ell_5 = \ell_1$ and so on). Thus C_2 and C_3 belong to the segment B_1B_4, C_1 lies on B_3B_4, and C_4 lies on B_1B_2. Assume that there exists an index i such that the area of $B_iB_{i+1}C_iC_{i+1}$ is at least $[P']/2$; then we have

$$\frac{[P']}{2} \leq [B_iB_{i+1}C_iC_{i+1}] = \frac{B_iC_i \cdot B_{i+1}C_{i+1} \cdot \sin\alpha_i}{2} < \frac{a_ia_{i+1}\sin\alpha_i}{2} \leq [P],$$

as desired. Assume, to the contrary, that such index does not exist. Two cases are possible.





Case 1. Assume that the rays B_1B_2 and B_4B_3 do not intersect (see the left figure above). This means, in particular, that $d(B_1, B_3B_4) \leq d(B_2, B_3B_4)$.

Since the ray B_3O lies in the angle $B_1B_3B_4$, we obtain $d(B_1, B_3C_3) \leq d(C_4, B_3C_3)$; hence $[B_3B_4B_1] \leq [B_3B_4C_3C_4] < [P']/2$. Similarly, $[B_1B_2B_4] \leq [B_1B_2C_1C_2] < [P']/2$. Thus,

$$\begin{aligned}[B_2B_3C_2C_3] &= [P'] - [B_1B_2C_3] - [B_3B_4C_2] = [P'] - \frac{B_1C_3}{B_1B_4}\cdot[B_1B_2B_4] - \frac{B_4C_2}{B_1B_4}\cdot[B_3B_4B_1] \\ &> [P']\left(1 - \frac{B_1C_3 + B_4C_2}{2B_1B_4}\right) \geq \frac{[P']}{2}.\end{aligned}$$

A contradiction.

Case 2. Assume now that the rays B_1B_2 and B_4B_3 intersect at some point (see the right figure above). Denote by L the common point of B_2C_1 and B_3C_4. We have $[B_2C_4C_1] \geq [B_2C_4B_3]$, hence $[C_1C_4L] \geq [B_2B_3L]$. Thus we have

$$\begin{aligned}[P'] > [B_1B_2C_1C_2] + [B_3B_4C_3C_4] &= [P'] + [LC_1C_2C_3C_4] - [B_2B_3L] \\ &\geq [P'] + [C_1C_4L] - [B_2B_3L] \geq [P'].\end{aligned}$$

A final contradiction.

Problem 5. Given an integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest $x > a_{n-1}$ such that:

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor .$$

Prove that every prime occurs in the sequence $a_1, a_2, \ldots$.

(Bulgaria) Alexander Ivanov

Solution 1. We prove that the a_n are precisely the *kth-power-free* positive integers, that is, those divisible by the kth power of no prime. The conclusion then follows.

Let B denote the set of all kth-power-free positive integers. We first show that, given a positive integer c,

$$\sum_{b \in B,\, b \leq c} \left\lfloor \sqrt[k]{\frac{c}{b}} \right\rfloor = c.$$

To this end, notice that every positive integer has a unique representation as a product of an element in B and a kth power. Consequently, the set of all positive integers less than or equal to c splits into

$$C_b = \{x \colon x \in \mathbb{Z}_{>0},\ x \leq c, \text{ and } x/b \text{ is a } k\text{th power}\}, \quad b \in B,\ b \leq c.$$

Clearly, $|C_b| = \left\lfloor \sqrt[k]{c/b} \right\rfloor$, whence the desired equality.

Finally, enumerate B according to the natural order: $1 = b_1 < b_2 < \cdots < b_n < \cdots$. We prove by induction on n that $a_n = b_n$. Clearly, $a_1 = b_1 = 1$, so let $n \geq 2$ and assume $a_m = b_m$ for all indices $m < n$. Since $b_n > b_{n-1} = a_{n-1}$ and

$$b_n = \sum_{i=1}^{n} \left\lfloor \sqrt[k]{\frac{b_n}{b_i}} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{b_n}{b_i}} \right\rfloor + 1 = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{b_n}{a_i}} \right\rfloor + 1,$$

the definition of a_n forces $a_n \leq b_n$. Were $a_n < b_n$, a contradiction would follow:

$$a_n = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{a_n}{b_i}} \right\rfloor = \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{a_n}{a_i}} \right\rfloor = a_n - 1.$$

Consequently, $a_n = b_n$. This completes the proof.

Solution 2. (*Ilya Bogdanov*) For every $n = 1, 2, 3, \ldots$, introduce the function

$$f_n(x) = x - 1 - \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor .$$

Denote also by $g_n(x)$ the number of the indices $i \leq n$ such that x/a_i is the kth power of an integer. Then $f_n(x+1) - f_n(x) = 1 - g_n(x)$ for every integer $x \geq a_n$; hence $f_n(x)+1 \geq f_n(x+1)$. Moreover, $f_n(a_{n-1}) = -1$ (since $f_{n-1}(a_{n-1}) = 0$). Now a straightforward induction shows that $f_n(x) < 0$ for all integers $x \in [a_{n-1}, a_n)$.

Next, if $g_n(x) > 0$ then $f_n(x) \leq f_n(x-1)$; this means that such an x cannot equal a_n. Thus a_j/a_i is never the kth power of an integer if $j > i$.

Now we are prepared to prove by induction on n that $a_1, a_2, \ldots, a_n$ are exactly all kth-power-free integers in $[1, a_n]$. The base case $n = 1$ is trivial.

Assume that all the kth-power-free integers on $[1, a_n]$ are exactly $a_1, \ldots, a_n$. Let b be the least integer larger than a_n such that $g_n(b) = 0$. We claim that: **(1)** $b = a_{n+1}$; and **(2)** b is the least kth-power-free number greater than a_n.

To prove **(1)**, notice first that all the numbers of the form a_j/a_i with $1 \le i < j \le n$ are not kth powers of *rational* numbers since a_i and a_j are kth-power-free. This means that for every integer $x \in (a_n, b)$ there exists exactly one index $i \le n$ such that x/a_i is the kth power of an integer (certainly, x is not kth-power-free). Hence $f_{n+1}(x) = f_{n+1}(x-1)$ for each such x, so $f_{n+1}(b-1) = f_{n+1}(a_n) = -1$. Next, since b/a_i is not the kth power of an integer, we have $f_{n+1}(b) = f_{n+1}(b-1) + 1 = 0$, thus $b = a_{n+1}$. This establishes **(1)**.

Finally, since all integers in (a_n, b) are not kth-power-free, we are left to prove that b is kth-power-free to establish **(2)**. Otherwise, let $y > 1$ be the greatest integer such that $y^k \mid b$; then b/y^k is kth-power-free and hence $b/y^k = a_i$ for some $i \le n$. So b/a_i is the kth power of an integer, which contradicts the definition of b.

Thus $a_1, a_2, \ldots$ are exactly all kth-power-free positive integers; consequently all primes are contained in this sequence.

Problem 6. $2n$ distinct tokens are placed at the vertices of a regular $2n$-gon, with one token placed at each vertex. A *move* consists of choosing an edge of the $2n$-gon and interchanging the two tokens at the endpoints of that edge. Suppose that after a finite number of moves, every pair of tokens have been interchanged exactly once. Prove that some edge has never been chosen.

(Russia) Alexander Gribalko

Solution. Step 1. Enumerate all the tokens in the initial arrangement in clockwise circular order; also enumerate the vertices of the $2n$-gon accordingly. Consider any three tokens $i < j < k$. At each moment, their cyclic order may be either i, j, k or i, k, j, counted clockwise. This order changes exactly when two of these three tokens have been switched. Hence the order has been reversed thrice, and in the final arrangement token k stands on the arc passing clockwise from token i to token j. Thus, at the end, token $i + 1$ is a counter-clockwise neighbor of token i for all $i = 1, 2, \dots, 2n - 1$, so the tokens in the final arrangement are numbered successively in counter-clockwise circular order.

This means that the final arrangement of tokens can be obtained from the initial one by reflection in some line ℓ.

Step 2. Notice that each token was involved into $2n-1$ switchings, so its initial and final vertices have different parity. Hence ℓ passes through the midpoints of two opposite sides of a $2n$-gon; we may assume that these are the sides a and b connecting $2n$ with 1 and n with $n+1$, respectively.

During the process, each token x has crossed ℓ at least once; thus one of its switchings has been made at edge a or at edge b. Assume that some two its switchings were performed at a and at b; we may (and will) assume that the one at a was earlier, and $x \leq n$. Then the total movement of token x consisted at least of: (i) moving from vertex x to a and crossing ℓ along a; (ii) moving from a to b and crossing ℓ along b; (iii) coming to vertex $2n + 1 - x$. This tales at least $x + n + (n - x) = 2n$ switchings, which is impossible.

Thus, each token had a switching at exactly one of the edges a and b.

Step 3. Finally, let us show that either each token has been switched at a, or each token has been switched at b (then the other edge has never been used, as desired). To the contrary, assume that there were switchings at both a and at b. Consider the first such switchings, and let x and y be the tokens which were moved clockwise during these switchings and crossed ℓ at a and b, respectively. By Step 2, $x \neq y$. Then tokens x and y initially were on opposite sides of ℓ.

Now consider the switching of tokens x and y; there was exactly one such switching, and we assume that it has been made on the same side of ℓ as vertex y. Then this switching has been made after token x had traced a. From this point on, token x is on the clockwise arc from token y to b, and it has no way to leave out from this arc. But this is impossible, since token y should trace b after that moment. A contradiction.

Remark. The same statement for $(2n-1)$-gon is also valid. The problem is stated for a polygon with an even number of sides only to avoid case consideration.

Let us outline the solution in the case of a $(2n - 1)$-gon. We prove the existence of line ℓ as in Step 1. This line passes through some vertex x, and through the midpoint of the opposite edge a. Then each token either passes through x, or crosses ℓ along a (but not both; this can be shown as in Step 2). Finally, since a token is involved into an even number of moves, it passes through x but not through a, and a is never used.