The 6th Romanian Master of Mathematics Competition

Day 2: Saturday, March 2, 2013, Bucharest

Language: English

Problem 4. Let P and P' be two convex quadrilateral regions in the plane (regions contain their boundary). Let them intersect, with O a point in the intersection. Suppose that for every line ℓ through O the segment $\ell \cap P$ is strictly longer than the segment $\ell \cap P'$. Is it possible that the ratio of the area of P' to the area of P is greater than 1.9?

Problem 5. Given an integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let a_n be the smallest $x > a_{n-1}$ such that:

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor .$$

Prove that every prime occurs in the sequence $a_1, a_2, \ldots$.

Problem 6. $2n$ distinct tokens are placed at the vertices of a regular $2n$-gon, with one token placed at each vertex. A *move* consists of choosing an edge of the $2n$-gon and interchanging the two tokens at the endpoints of that edge. Suppose that after a finite number of moves, every pair of tokens have been interchanged exactly once. Prove that some edge has never been chosen.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.