The 9th Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. **(a)** Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_{2k+1}$ are integers.
This number k is called the *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

Vjekoslav Kovač, Croatia

Solution. **(a)** We show by induction on the integer $M \geq 0$ that every integer n in the range $-2^M + 1$ through 2^M can uniquely be written in the form $n = \sum_{j=1}^{\ell} (-1)^{j-1} 2^{m_j}$ for some integers $\ell \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_\ell \leq M$ (empty sums are 0); moreover, in this unique representation ℓ is odd if $n > 0$, and even if $n \leq 0$. The integer $w(n) = \lfloor \ell/2 \rfloor$ is called the *weight* of n.

Existence once proved, uniqueness follows from the fact that there are as many such representations as integers in the range $-2^M + 1$ through 2^M, namely, 2^{M+1}.

To prove existence, notice that the base case $M = 0$ is clear, so let $M \geq 1$ and let n be an integer in the range $-2^M + 1$ through 2^M.

If $-2^M + 1 \leq n \leq -2^{M-1}$, then $1 \leq n + 2^M \leq 2^{M-1}$, so $n + 2^M = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j}$ for some integers $k \geq 0$ and $0 \leq m_1 < \cdots < m_{2k+1} \leq M - 1$ by the induction hypothesis, and $n = \sum_{j=1}^{2k+2} (-1)^{j-1} 2^{m_j}$, where $m_{2k+2} = M$.

The case $-2^{M-1} + 1 \leq n \leq 2^{M-1}$ is covered by the induction hypothesis.

Finally, if $2^{M-1} + 1 \leq n \leq 2^M$, then $-2^{M-1} + 1 \leq n - 2^M \leq 0$, so $n - 2^M = \sum_{j=1}^{2k} (-1)^{j-1} 2^{m_j}$ for some integers $k \geq 0$ and $0 \leq m_1 < \cdots < m_{2k} \leq M - 1$ by the induction hypothesis, and $n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j}$, where $m_{2k+1} = M$.

(b) First Approach. Let $M \geq 0$ be an integer. The solution for part **(a)** shows that the number of even (respectively, odd) weight integers in the range 1 through 2^M coincides with the number of subsets in $\{0, 1, 2, \ldots, M\}$ whose cardinality has remainder 1 (respectively, 3) modulo 4. Therefore, the difference of these numbers is

$$\sum_{k=0}^{\lfloor M/2 \rfloor} (-1)^k \binom{M+1}{2k+1} = \frac{(1+\mathrm{i})^{M+1} - (1-\mathrm{i})^{M+1}}{2\mathrm{i}} = 2^{(M+1)/2} \sin \frac{(M+1)\pi}{4},$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Thus, the required difference is 2^{1009}.

Second Approach. For every integer $M \geq 0$, let $A_M = \sum_{n=-2^M+1}^{0} (-1)^{w(n)}$ and let $B_M = \sum_{n=1}^{2^M} (-1)^{w(n)}$; thus, B_M evaluates the difference of the number of even weight integers in the range 1 through 2^M and the number of odd weight integers in that range.

Notice that

$$w(n) = \begin{cases} w(n + 2^M) + 1 & \text{if } -2^M + 1 \leq n \leq -2^{M-1}, \\ w(n - 2^M) & \text{if } 2^{M-1} + 1 \leq n \leq 2^M, \end{cases}$$

to get

$$A_M = -\sum_{n=-2^M+1}^{-2^{M-1}} (-1)^{w(n+2^M)} + \sum_{n=-2^{M-1}+1}^{0} (-1)^{w(n)} = -B_{M-1} + A_{M-1},$$

$$B_M = \sum_{n=1}^{2^{M-1}} (-1)^{w(n)} + \sum_{n=2^{M-1}+1}^{2^M} (-1)^{w(n-2^M)} = B_{M-1} + A_{M-1}.$$

Iteration yields

$$\begin{aligned} B_M &= A_{M-1} + B_{M-1} = (A_{M-2} - B_{M-2}) + (A_{M-2} + B_{M-2}) = 2A_{M-2} \\ &= 2A_{M-3} - 2B_{M-3} = 2(A_{M-4} - B_{M-4}) - 2(A_{M-4} + B_{M-4}) = -4B_{M-4}. \end{aligned}$$

Thus, $B_{2017} = (-4)^{504} B_1 = 2^{1008} B_1$; since $B_1 = (-1)^{w(1)} + (-1)^{w(2)} = 2$, it follows that $B_{2017} = 2^{1009}$.

Problem 2. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$, and $k+1$ distinct integers $x_1, x_2, \ldots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1}).$$

Semen Petrov, Russia

Note. A polynomial is *monic* if the coefficient of the highest power is one.

Solution. There is only one such integer, namely, $n = 2$. In this case, if P is a constant polynomial, the required condition is clearly satisfied; if $P = X + c$, then $P(c-1) + P(c+1) = P(3c)$; and if $P = X^2 + qX + r$, then $P(X) = P(-X-q)$.

To rule out all other values of n, it is sufficient to exhibit a monic polynomial P of degree at most n with integer coefficients, whose restriction to the integers is injective, and $P(x) \equiv 1 \pmod{n}$ for all integers x. This is easily seen by reading the relation in the statement modulo n, to deduce that $k \equiv 1 \pmod{n}$, so $k = 1$, since $1 \leq k \leq n$; hence $P(x_1) = P(x_2)$ for some distinct integers x_1 and x_2, which contradicts injectivity.

If $n = 1$, let $P = X$, and if $n = 4$, let $P = X^4 + 7X^2 + 4X + 1$. In the latter case, clearly, $P(x) \equiv 1 \pmod{4}$ for all integers x; and P is injective on the integers, since $P(x) - P(y) = (x-y)\big((x+y)(x^2+y^2+7)+4\big)$, and the absolute value of $(x+y)(x^2+y^2+7)$ is either 0 or at least 7 for integral x and y.

Assume henceforth $n \geq 3$, $n \neq 4$, and let $f_n = (X-1)(X-2)\cdots(X-n)$. Clearly, $f_n(x) \equiv 0 \pmod{n}$ for all integers x. If n is odd, then f_n is non-decreasing on the integers; and if, in addition, $n > 3$, then $f_n(x) \equiv 0 \pmod{n+1}$ for all integers x, since $f_n(0) = -n! = -1 \cdot 2 \cdot \cdots \cdot \frac{n+1}{2} \cdot \cdots \cdot n \equiv 0 \pmod{n+1}$.

Finally, let $P = f_n + nX + 1$ if n is odd, and let $P = f_{n-1} + nX + 1$ if n is even. In either case, P is strictly increasing, hence injective, on the integers, and $P(x) \equiv 1 \pmod{n}$ for all integers x.

Remark. The polynomial $P = f_n + nX + 1$ works equally well for even $n > 2$. To prove injectivity, notice that P is strictly monotone, hence injective, on non-positive (respectively, positive) integers. Suppose, if possible, that $P(a) = P(b)$ for some integers $a \leq 0$ and $b > 0$. Notice that $P(a) \geq P(0) = n! + 1 > n^2 + 1 = P(n)$, since $n \geq 4$, to infer that $b \geq n+1$. It is therefore sufficient to show that $P(x) > P(n+1-x) > P(x-1)$ for all integers $x \geq n+1$. The former inequality is trivial, since $f_n(x) = f_n(n+1-x)$ for even n. For the latter, write

$$\begin{aligned} P(n+1-x) - P(x-1) &= (x-1)\cdots(x-n) - (x-2)\cdots(x-n-1) + n(n+2-2x) \\ &= n\big((x-2)\cdots(x-n) + (n-2) - 2(x-2)\big) \geq n(n-2) > 0, \end{aligned}$$

since $(x-3)\cdots(x-n) \geq 2$.

Problem 3. Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, \ldots, A_k$ of X is *tight* if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

Alexander Polyansky, Russia

Solution 1. (*Ilya Bogdanov*) The required maximum is $2n-2$. To describe a $(2n-2)$-element collection satisfying the required conditions, write $X = \{1, 2, \ldots, n\}$ and set $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, n-1$, and $B_k = \{k-n+2, k-n+3, \ldots, n\}$, $k = n, n+1, \ldots, 2n-2$. To show that no subcollection of the B_k is tight, consider a subcollection $\mathcal{C}$ whose union U is a proper subset of X, let m be an element in $X \setminus U$, and notice that $\mathcal{C}$ is a subcollection of $\{B_1, \ldots, B_{m-1}, B_{m+n-1}, \ldots, B_{2n-2}\}$, since the other B's are precisely those containing m. If U contains elements less than m, let k be the greatest such and notice that B_k is the only member of $\mathcal{C}$ containing k; and if U contains elements greater than m, let k be the least such and notice that B_{k+n-2} is the only member of $\mathcal{C}$ containing k. Consequently, $\mathcal{C}$ is not tight.

We now proceed to show by induction on $n \geq 2$ that the cardinality of a collection of proper non-empty subsets of X, no subcollection of which is tight, does not exceed $2n-2$. The base case $n = 2$ is clear, so let $n > 2$ and suppose, if possible, that $\mathcal{B}$ is a collection of $2n-1$ proper non-empty subsets of X containing no tight subcollection.

To begin, notice that $\mathcal{B}$ has an empty intersection: if the members of $\mathcal{B}$ shared an element x, then $\mathcal{B}' = \{B \setminus \{x\} \colon B \in \mathcal{B}, B \neq \{x\}\}$ would be a collection of at least $2n-2$ proper non-empty subsets of $X \setminus \{x\}$ containing no tight subcollection, and the induction hypothesis would be contradicted.

Now, for every x in X, let $\mathcal{B}_x$ be the (non-empty) collection of all members of $\mathcal{B}$ not containing x. Since no subcollection of $\mathcal{B}$ is tight, $\mathcal{B}_x$ is not tight, and since the union of $\mathcal{B}_x$ does not contain x, some x' in X is covered by a single member of $\mathcal{B}_x$. In other words, *there is a single set in $\mathcal{B}$ covering x' but not x*. In this case, draw an arrow from x to x'. Since there is at least one arrow from each x in X, some of these arrows form a (minimal) cycle $x_1 \to x_2 \to \cdots \to x_k \to x_{k+1} = x_1$ for some suitable integer $k \geq 2$. Let A_i be the unique member of $\mathcal{B}$ containing x_{i+1} but not x_i, and let $X' = \{x_1, x_2, \ldots, x_k\}$.

Remove $A_1, A_2, \ldots, A_k$ from $\mathcal{B}$ to obtain a collection $\mathcal{B}'$ each member of which either contains or is disjoint from X': for if a member B of $\mathcal{B}'$ contained some but not all elements of X', then B should contain x_{i+1} but not x_i for some i, and $B = A_i$, a contradiction. This rules out the case $k = n$, for otherwise $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$, so $|\mathcal{B}| < 2n-1$.

To rule out the case $k < n$, consider an extra element x^* outside X and let

$$\mathcal{B}^* = \{B \colon B \in \mathcal{B}', \ B \cap X' = \varnothing\} \cup \{(B \setminus X') \cup \{x^*\} \colon B \in \mathcal{B}', \ X' \subseteq B\};$$

thus, in each member of $\mathcal{B}'$ containing X', the latter is collapsed to singleton x^*. Notice that $\mathcal{B}^*$ is a collection of proper non-empty subsets of $X^* = (X \setminus X') \cup \{x^*\}$, no subcollection of which is tight. By the induction hypothesis, $|\mathcal{B}'| = |\mathcal{B}^*| \leq 2|X^*| - 2 = 2(n-k)$, so $|\mathcal{B}| \leq 2(n-k) + k = 2n - k < 2n - 1$, a final contradiction.

Solution 2. Proceed again by induction on n to show that the cardinality of a collection of proper non-empty subsets of X, no subcollection of which is tight, does not exceed $2n-2$.

Consider any collection $\mathcal{B}$ of proper non-empty subsets of X with no tight subcollection (we call such collection *good*). Assume that there exist $M, N \in \mathcal{B}$ such that $M \cup N$ is distinct from M, N, and X. In this case, we will show how to modify $\mathcal{B}$ so that it remains good, contains the same number of sets, but the total number of elements in the sets of $\mathcal{B}$ increases.

Consider a maximal (relative to set-theoretic inclusion) subcollection $\mathcal{C} \subseteq \mathcal{B}$ such that the set $C = \bigcup_{A \in \mathcal{C}} A$ is distinct from X and from all members of $\mathcal{C}$. Notice here that the union of *any* subcollection $\mathcal{D} \subset \mathcal{B}$ cannot coincide with any $K \in \mathcal{B} \setminus \mathcal{D}$, otherwise $\{K\} \cup \mathcal{D}$ would be tight. Surely, $\mathcal{C}$ exists (since $\{M, N\}$ is an example of a collection satisfying the requirements on $\mathcal{C}$, except for maximality); moreover, $C \notin \mathcal{B}$ by the above remark.

Since $C \neq X$, there exists an $L \in \mathcal{C}$ and $x \in L$ such that L is the unique set in $\mathcal{C}$ containing x. Now replace in $\mathcal{B}$ the set L by C in order to obtain a new collection $\mathcal{B}'$ (then $|\mathcal{B}'| = |\mathcal{B}|$). We claim that $\mathcal{B}'$ is good.

Assume, to the contrary, that $\mathcal{B}'$ contained a tight subcollection $\mathcal{T}$; clearly, $C \in \mathcal{T}$, otherwise $\mathcal{B}$ is not good. If $\mathcal{T} \subseteq \mathcal{C} \cup \{C\}$, then C is the unique set in $\mathcal{T}$ containing x which is impossible. Therefore, there exists $P \in \mathcal{T} \setminus (\mathcal{C} \cup \{C\})$. By maximality of $\mathcal{C}$, the collection $\mathcal{C} \cup \{P\}$ does not satisfy the requirements imposed on $\mathcal{C}$; since $P \cup C \neq X$, this may happen only if $C \cup P = P$, i.e., if $C \subset P$. But then $\mathcal{G} = (\mathcal{T} \setminus \{C\}) \cup \mathcal{C}$ is a tight subcollection in $\mathcal{B}$: all elements of C are covered by $\mathcal{G}$ at least twice (by P and an element of $\mathcal{C}$), and all the rest elements are covered by $\mathcal{G}$ the same number of times as by $\mathcal{T}$. A contradiction. Thus $\mathcal{B}'$ is good.

Such modifications may be performed finitely many times, since the total number of elements of sets in $\mathcal{B}$ increases. Thus, at some moment we arrive at a good collection $\mathcal{B}$ for which the procedure no longer applies. This means that for every $M, N \in \mathcal{B}$, either $M \cup N = X$ or one of them is contained in the other.

Now let M be a minimal (with respect to inclusion) set in $\mathcal{B}$. Then each set in $\mathcal{B}$ either contains M or forms X in union with M (i.e., contains $X \setminus M$). Now one may easily see that the two collections

$$\mathcal{B}_+ = \{A \setminus M : A \in \mathcal{B},\ M \subset A,\ A \neq M\}, \quad \mathcal{B}_- = \{A \cap M : A \in \mathcal{B},\ X \setminus M \subset A,\ A \neq X \setminus M\}$$

are good as collections of subsets of $X \setminus M$ and M, respectively; thus, by the induction hypothesis, we have $|\mathcal{B}_+| + |\mathcal{B}_-| \leq 2n - 4$.

Finally, each set $A \in \mathcal{B}$ either produces a set in one of the two new collections, or coincides with M or $X \setminus M$. Thus $|\mathcal{B}| \leq |\mathcal{B}_+| + |\mathcal{B}_-| + 2 \leq 2n - 2$, as required.

Solution 3. We provide yet another proof of the estimate $|\mathcal{B}| \leq 2n - 2$, using the notion of a *good* collection from Solution 2. Arguing indirectly, we assume that there exists a good collection $\mathcal{B}$ with $|\mathcal{B}| \geq 2n - 1$, and choose one such for the minimal possible value of n. Clearly, $n > 2$.

Firstly, we perform a different modification of $\mathcal{B}$. Choose any $x \in X$, and consider the subcollection $\mathcal{B}_x = \{B : B \in \mathcal{B},\ x \notin B\}$. By our assumption, $\mathcal{B}_x$ is not tight. As the union of sets in $\mathcal{B}_x$ is distinct from X, either this collection is empty, or there exists an element $y \in X$ contained in a unique member A_x of $\mathcal{B}_x$. In the former case, we add the set $B_x = X \setminus \{x\}$ to $\mathcal{B}$, and in the latter we replace A_x by B_x, to form a new collection $\mathcal{B}'$. (Notice that if $B_x \in \mathcal{B}$, then $B_x \in \mathcal{B}_x$ and $y \in B_x$, so $B_x = A_x$.)

We claim that the collection $\mathcal{B}'$ is also good. Indeed, if $\mathcal{B}'$ has a tight subcollection $\mathcal{T}$, then B_x should lie in $\mathcal{T}$. Then, as the union of the sets in $\mathcal{T}$ is distinct from X, we should have $\mathcal{T} \subseteq \mathcal{B}_x \cup \{B_x\}$. But in this case an element y is contained in a unique member of $\mathcal{T}$, namely B_x, so $\mathcal{T}$ is not tight — a contradiction.

Perform this procedure for every $x \in X$, to get a good collection $\mathcal{B}$ containing the sets $B_x = X \setminus \{x\}$ for all $x \in X$. Consider now an element $x \in X$ such that $|\mathcal{B}_x|$ is maximal. As we have mentioned before, there exists an element $y \in X$ belonging to a unique member (namely, B_x) of $\mathcal{B}_x$. Thus, $\mathcal{B}_x \setminus \{B_x\} \subset \mathcal{B}_y$; also, $B_y \in \mathcal{B}_y \setminus \mathcal{B}_x$. Thus we get $|\mathcal{B}_y| \geq |\mathcal{B}_x|$, which by the maximality assumption yields the equality, which in turn means that $\mathcal{B}_y = (\mathcal{B}_x \setminus \{B_x\}) \cup \{B_y\}$.

Therefore, each set in $\mathcal{B} \setminus \{B_x, B_y\}$ contains either both x and y, or none of them. Collapsing $\{x, y\}$ to singleton x^*, we get a new collection of $|\mathcal{B}| - 2$ subsets of $(X \setminus \{x, y\}) \cup \{x^*\}$ containing no tight subcollection. This contradicts minimality of n.

Remarks. 1. Removal of the condition that subsets be proper would only increase the maximum by 1. The 'non-emptiness' condition could also be omitted, since the empty set forms a tight collection by itself, but the argument is a bit too formal to be considered.

2. There are many different examples of good collections of $2n-2$ sets. E.g., applying the algorithm from the first part of Solution 2 to the example shown in Solution 1, one may get the following example: $B_k = \{1, 2, \ldots, k\}$, $k = 1, 2, \ldots, n-1$, and $B_k = X \setminus \{k-n+1\}$, $k = n, n+1, \ldots, 2n-2$.