The 10^{th} Romanian Master of Mathematics Competition

Day 1 — Solutions

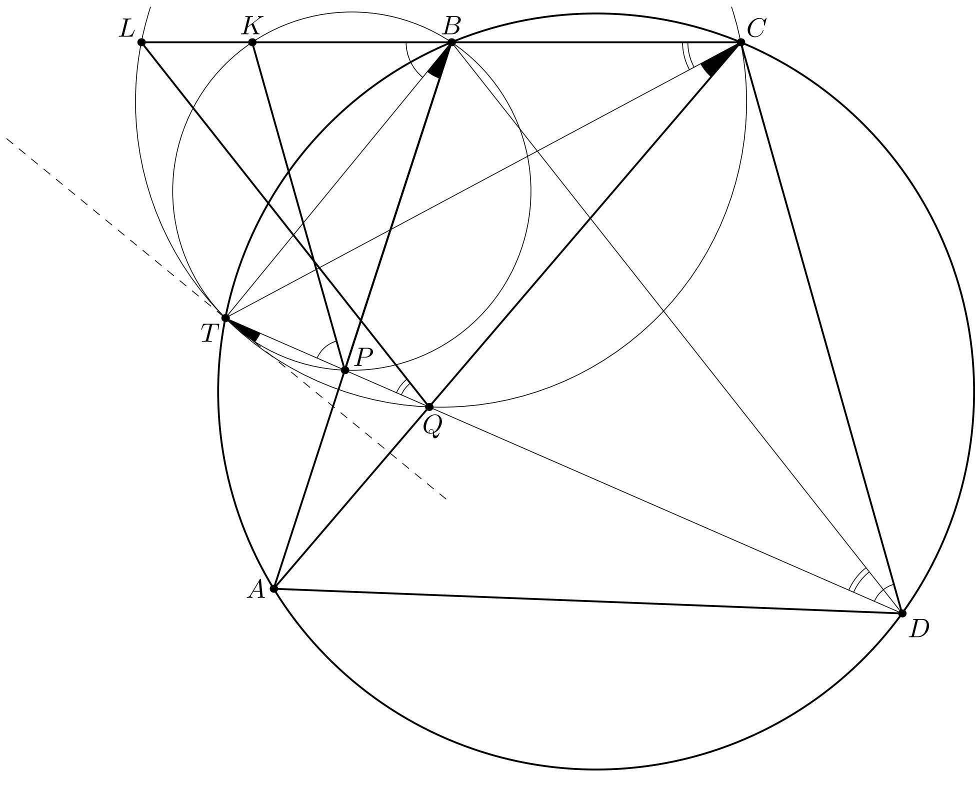
Problem 1. Let $ABCD$ be a cyclic quadrangle and let P be a point on the side AB. The diagonal AC crosses the segment DP at Q. The parallel through P to CD crosses the extension of the side BC beyond B at K, and the parallel through Q to BD crosses the extension of the side BC beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Aleksandr Kuznetsov, Russia

Solution. We show that the circles BKP and CLQ are tangent at the point T where the line DP crosses the circle $ABCD$ again.

Since $BCDT$ is cyclic, we have $\angle KBT = \angle CDT$. Since $KP \parallel CD$, we get $\angle CDT = \angle KPT$. Thus, $\angle KBT = \angle CDT = \angle KPT$, which shows that T lies on the circle BKP. Similarly, the equalities $\angle LCT = \angle BDT = \angle LQT$ show that T also lies on the circle CLQ.

It remains to prove that these circles are indeed tangent at T. This follows from the fact that the chords TP and TQ in the circles $BKTP$ and $CLTQ$, respectively, both lie along the same line and subtend equal angles $\angle TBP = \angle TBA = \angle TCA = \angle TCQ$.



Remarks. The point T may alternatively be defined as the Miquel point of (any four of) the five lines AB, BC, AC, KP, and LQ.

Of course, the result still holds if P is chosen on the line AB, and the other points lie on the corresponding lines rather than segments/rays. The current formulation was chosen in order to avoid case distinction based on the possible configurations of points.

Problem 2. Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}. \tag{$*$}$$

Ilya Bogdanov, Russia

Solution 1. The answer is in the negative. Comparing the degrees of both sides in $(*)$ we get $\deg P = 21n$ and $\deg Q = 10n$ for some positive integer n. Take the derivative of $(*)$ to obtain

$$P'P^8(10P + 9) = Q'Q^{19}(21Q + 20). \tag{$**$}$$

Since $\gcd(10P+9, P) = \gcd(10P+9, P+1) = 1$, it follows that $\gcd(10P+9, P^9(P+1)) = 1$, so $\gcd(10P + 9, Q) = 1$, by $(*)$. Thus $(**)$ yields $10P + 9 \mid Q'(21Q + 20)$, which is impossible since $0 < \deg(Q'(21Q + 20)) = 20n - 1 < 21n = \deg(10P + 9)$. A contradiction.

Remark. A similar argument shows that there are no non-constant solutions of $P^m + P^{m-1} = Q^k + Q^{k-1}$, where k and m are positive integers with $k \geq 2m$. A critical case is $k = 2m$; but in this case there exist more routine ways of solving the problem. Thus, we decided to choose $k = 2m + 1$.

Solution 2. Letting r and s be integers such that $r \geq 2$ and $s \geq 2r$, we show that if $P^r + P^{r-1} = Q^s + Q^{s-1}$, then Q is constant.

Let $m = \deg P$ and $n = \deg Q$. A degree inspection in the given relation shows that $m \geq 2n$.

We will prove that $P(P + 1)$ has at least $m + 1$ distinct complex roots. Assuming this for the moment, notice that Q takes on one of the values 0 or -1 at each of those roots. Since $m + 1 \geq 2n + 1$, it follows that Q takes on one of the values 0 and -1 at more than n distinct points, so Q must be constant.

Finally, we prove that $P(P + 1)$ has at least $m + 1$ distinct complex roots. This can be done either by referring to the Mason–Stothers theorem or directly, in terms of multiplicities of the roots in question.

Since P and $P+1$ are relatively prime, the Mason–Stothers theorem implies that the number of distinct roots of $P(P + 1)$ is greater than m, hence at least $m + 1$.

For a direct proof, let $z_1, \ldots, z_t$ be the distinct complex roots of $P(P + 1)$, and let z_k have multiplicity α_k, $k = 1, \ldots, t$. Since P and $P + 1$ have no roots in common, and $P' = (P + 1)'$, it follows that P' has a root of multiplicity $\alpha_k - 1$ at z_k. Consequently, $m - 1 = \deg P' \geq \sum_{k=1}^{t}(\alpha_k - 1) = \sum_{k=1}^{t} \alpha_k - t = 2m - t$; that is, $t \geq m + 1$. This completes the prof.

Remark. The Mason–Stothers theorem (in a particular case over the complex field) claims that, given coprime complex polynomials $P(x)$, $Q(x)$, and $R(x)$, not all constant, such that $P(x) + Q(x) = R(x)$, the total number of their complex roots (**not** regarding multiplicities) is at least $\max\{\deg P, \deg Q, \deg R\} + 1$. This theorem was a part of motivation for the famous *abc*-conjecture.

Problem 3. Ann and Bob play a game on an infinite checkered plane making moves in turn; Ann makes the first move. A move consists in orienting any unit grid-segment that has not been oriented before. If at some stage some oriented segments form an oriented cycle, Bob wins. Does Bob have a strategy that guarantees him to win?

Maxim Didin, Russia

Solution. The answer is in the negative: Ann has a strategy allowing her to prevent Bob's victory.

We say that two unit grid-segments form a *low-left corner* (or *LL-corner*) if they share an endpoint which is the lowest point of one and the leftmost point of the other. An *up-right corner* (or *UR-corner*) is defined similarly. The common endpoint of two unit grid-segments at a corner is the *joint* of that corner.

Fix a vertical line on the grid and call it the *midline*; the unit grid-segments along the midline are called *middle segments*. The unit grid-segments lying to the left/right of the midline are called *left/right segments*. Partition all left segments into LL-corners, and all right segments into UR-corners.

We now describe Ann's strategy. Her first move consists in orienting some middle segment arbitrarily. Assume that at some stage, Bob orients some segment s. If s is a middle segment, Ann orients any free middle segment arbitrarily. Otherwise, s forms a corner in the partition with some other segment t. Then Ann orients t so that the joint of the corner is either the source of both arrows, or the target of both. Notice that after any move of Ann's, each corner in the partition is either completely oriented or completely not oriented. This means that Ann can always make a required move.

Assume that Bob wins at some stage, i.e., an oriented cycle C occurs. Let X be the lowest of the leftmost points of C, and let Y be the topmost of the rightmost points of C. If X lies (strictly) to the left of the midline, then X is the joint of some corner whose segments are both oriented. But, according to Ann's strategy, they are oriented so that they cannot occur in a cycle — a contradiction. Otherwise, Y lies to the right of the midline, and a similar argument applies. Thus, Bob will never win, as desired.

Remarks. (1) There are several variations of the argument in the solution above. For instance, instead of the midline, Ann may choose any infinite in both directions down going polyline along the grid (i.e., consisting of steps to the right and steps-down alone). Alternatively, she may split the plane into four quadrants, use their borders as "trash bin" (as the midline was used in the solution above), partition all segments in the upper-right quadrant into UR-corners, all segments in the lower-right quadrant into LR-corners, and so on.

(2) The problem becomes easier if Bob makes the first move. In this case, his opponent just partitions the whole grid into LL-corners. In particular, one may change the problem to say that the first player to achieve an oriented cycle wins (in this case, the result is a draw).

On the other hand, this version is closer to known problems. In particular, the following problem is known:

> *Ann and Bob play the game on an infinite checkered plane making moves in turn (Ann makes the first move). A move consists in painting any unit grid segment that has not been painted before (Ann paints in blue, Bob paints in red). If a player creates a cycle of her/his color, (s)he wins. Does any of the players have a winning strategy?*

Again, the solution is pairing strategy with corners of a fixed orientation (with a little twist for Ann's strategy — in this problem, it is clear that Ann has better chances).