The 13th Romanian Master of Mathematics Competition

Day 1 — Solutions

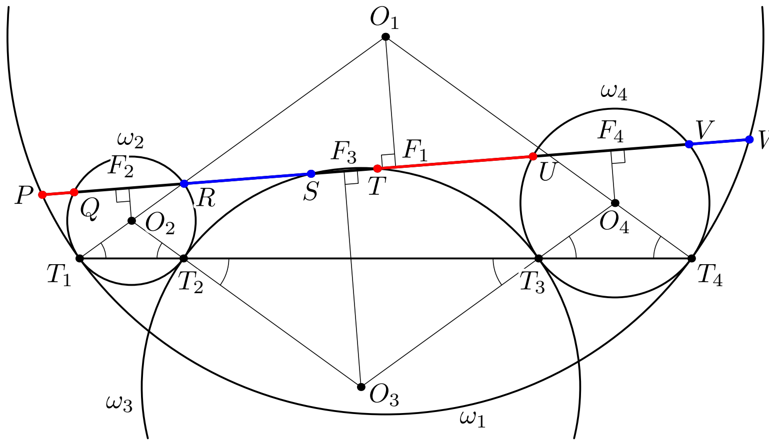
Problem 1. Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$.

Hungary, Geza Kos

Solution. Let O_i be the centre of ω_i, $i = 1, 2, 3, 4$. Notice that the isosceles triangles $O_iT_iT_{i-1}$ are similar (indices are reduced modulo 4), to infer that ω_4 is internally tangent to ω_1 at T_4, and $O_1O_2O_3O_4$ is a (possibly degenerate) parallelogram.

Let F_i be the foot of the perpendicular from O_i to PW. The F_i clearly bisect the segments PW, QR, ST and UV, respectively.

The proof can now be concluded in two similar ways.



First Approach. Since $O_1O_2O_3O_4$ is a parallelogram, $\overrightarrow{F_1F_2} + \overrightarrow{F_3F_4} = \mathbf{0}$ and $\overrightarrow{F_2F_3} + \overrightarrow{F_4F_1} = \mathbf{0}$; this still holds in the degenerate case, for if the O_i are collinear, then they all lie on the line T_1T_4, and each O_i is the midpoint of the segment T_iT_{i+1}. Consequently,

$$\begin{aligned}
\overrightarrow{PQ} - \overrightarrow{RS} + \overrightarrow{TU} - \overrightarrow{VW} &= \left(\overrightarrow{PF_1} + \overrightarrow{F_1F_2} + \overrightarrow{F_2Q}\right) - \left(\overrightarrow{RF_2} + \overrightarrow{F_2F_3} + \overrightarrow{F_3S}\right) \\
&\quad + \left(\overrightarrow{TF_3} + \overrightarrow{F_3F_4} + \overrightarrow{F_4U}\right) - \left(\overrightarrow{VF_4} + \overrightarrow{F_4F_1} + \overrightarrow{F_1W}\right) \\
&= \left(\overrightarrow{PF_1} - \overrightarrow{F_1W}\right) - \left(\overrightarrow{RF_2} - \overrightarrow{F_2Q}\right) + \left(\overrightarrow{TF_3} - \overrightarrow{F_3S}\right) - \left(\overrightarrow{VF_4} - \overrightarrow{F_4U}\right) \\
&\quad + \left(\overrightarrow{F_1F_2} + \overrightarrow{F_3F_4}\right) - \left(\overrightarrow{F_2F_3} + \overrightarrow{F_4F_1}\right) = \mathbf{0}.
\end{aligned}$$

Alternatively, but equivalently, $\overrightarrow{PQ} + \overrightarrow{TU} = \overrightarrow{RS} + \overrightarrow{VW}$, as required.

Second Approach. This is merely another way of reading the previous argument. Fix an orientation of the line PW, say, from P towards W, and use a lower case letter to denote the coordinate of a point labelled by the corresponding upper case letter.

Since the diagonals of a parallelogram bisect one another, $f_1 + f_3 = f_2 + f_4$, the common value being twice the coordinate of the projection to PW of the point where O_1O_3 and O_2O_4 cross; the relation clearly holds in the degenerate case as well.

Plug $f_1 = \frac{1}{2}(p+w)$, $f_2 = \frac{1}{2}(q+r)$, $f_3 = \frac{1}{2}(s+t)$ and $f_4 = \frac{1}{2}(u+v)$ into the above equality to get $p+w+s+t = q+r+u+v$. Alternatively, but equivalently, $(q-p)+(u-t) = (s-r)+(w-v)$, that is, $PQ + TU = RQ + VW$, as required.

Problem 2. Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers $a_1, a_2, \ldots, a_{20}$ such that, for each $k = 1, 2, \ldots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k \colon k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Russia, Sergey Kudrya

Solution. Sergey can determine Xenia's number in 2 but not fewer moves.

We first show that 2 moves are sufficient. Let Sergey provide the set $\{17, 18\}$ on his first move, and the set $\{18, 19\}$ on the second move. In Xenia's two responses, exactly one number occurs twice, namely, a_{18}. Thus, Sergey is able to identify a_{17}, a_{18}, and a_{19}, and thence the residue of N modulo $17 \cdot 18 \cdot 19 = 5814 > 5000$, by the Chinese Remainder Theorem. This means that the given range contains a single number satisfying all congruences, and Sergey achieves his goal.

To show that 1 move is not sufficient, let $M = \operatorname{lcm}(1, 2, \ldots, 10) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. Notice that M is divisible by the greatest common divisor of every pair of distinct positive integers not exceeding 20. Let Sergey provide the set $S = \{s_1, s_2, \ldots, s_k\}$. We show that there exist pairwise distinct positive integers $b_1, b_2, \ldots, b_k$ such that $1 \equiv b_i \pmod{s_i}$ and $M + 1 \equiv b_{i-1} \pmod{s_i}$ (indices are reduced modulo k). Thus, if in response Xenia provides the set $\{b_1, b_2, \ldots, b_k\}$, then Sergey will be unable to distinguish 1 from $M + 1$, as desired.

To this end, notice that, for each i, the numbers of the form $1 + ms_i$, $m \in \mathbb{Z}$, cover all residues modulo s_{i+1} which are congruent to $1 \, (\equiv M + 1)$ modulo $\gcd(s_i, s_{i+1}) \mid M$. Xenia can therefore choose a positive integer b_i such that $b_i \equiv 1 \pmod{s_i}$ and $b_i \equiv M + 1 \pmod{s_{i+1}}$. Clearly, such choices can be performed so as to make the b_i pairwise distinct, as required.

Problem 3. A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Russia, Mikhail Antipov

Solution. Assume that every single worker also forms a brigade (with a unique possible leader). In this modified setting, we are interested in the number N of ways to assign leadership so that each worker's number of assignments is congruent to 1 modulo 4.

Consider the variables x_1, x_2, $\ldots$, x_{17} corresponding to the workers. Assign each brigade (from the i-th through the j-th worker) the polynomial $f_{ij} = x_i + x_{i+1} + \cdots + x_j$, and form the product $f = \prod_{1 \le i \le j \le 17} f_{ij}$. The number N is the sum $\Sigma(f)$ of the coefficients of all monomials $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_{17}^{\alpha_{17}}$ in the expansion of f, where the α_i are all congruent to 1 modulo 4. For any polynomial P, let $\Sigma(P)$ denote the corresponding sum. From now on, all polynomials are considered with coefficients in the finite field $\mathbb{F}_{17}$.

Recall that for any positive integer n, and any integers a_1, a_2, $\ldots$, a_n, there exist indices $i \le j$ such that $a_i + a_{i+1} + \cdots + a_j$ is divisible by n. Consequently, $f(a_1, a_2, \ldots, a_{17}) = 0$ for all a_1, a_2, $\ldots$, a_{17} in $\mathbb{F}_{17}$.

Now, if some monomial in the expansion of f is divisible by x_i^{17}, replace that x_i^{17} by x_i; this does not alter the above overall vanishing property (by Fermat's Little Theorem), and preserves $\Sigma(f)$. After several such changes, f transforms into a polynomial g whose degree in each variable does not exceed 16, and $g(a_1, a_2, \ldots, a_{17}) = 0$ for all a_1, a_2, $\ldots$, a_{17} in $\mathbb{F}_{17}$. For such a polynomial, an easy induction on the number of variables shows that it is identically zero. Consequently, $\Sigma(g) = 0$, so $\Sigma(f) = 0$ as well, as desired.