The 13$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Consider an integer $n \geq 2$ and write the numbers 1, 2, …, n down on a board. A move consists in erasing any two numbers a and b, and, for each c in $\{a+b, |a-b|\}$, writing c down on the board, unless c is already there; if c is already on the board, do nothing. For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

China

Solution. The answer is in the affirmative for all $n \geq 2$. Induct on n. Leaving aside the trivial case $n = 2$, deal first with particular cases $n = 5$ and $n = 6$.

If $n = 5$, remove first the pair $(2, 5)$, notice that $3 = |2 - 5|$ is already on the board, so $7 = 2 + 5$ alone is written down. Removal of the pair $(3, 4)$ then leaves exactly two numbers on the board, 1 and 7, since $|3 \pm 4|$ are both already there.

If $n = 6$, remove first the pair $(1, 6)$, notice that $5 = |1 - 6|$ is already on the board, so $7 = 1 + 6$ alone is written down. Next, remove the pair $(2, 5)$ and notice that $|2 \pm 5|$ are both already on the board, so no new number is written down. Finally, removal of the pair $(3, 4)$ provides a single number to be written down, $1 = |3 - 4|$, since $7 = 3 + 4$ is already on the board. At this stage, the process comes to an end: 1 and 7 are the two numbers left.

In the remaining cases, the problem for n is brought down to the corresponding problem for $\lceil n/2 \rceil < n$ by a finite number of moves. The conclusion then follows by induction.

Let $n = 4k$ or $4k - 1$, where k is a positive integer. Remove the pairs $(1, 4k - 1)$, $(3, 4k - 3)$, …, $(2k - 1, 2k + 1)$ in turn. Each time, two odd numbers are removed, and the corresponding $c = |a \pm b|$ are even numbers in the range 2 through $4k$, of which one is always $4k$. These even numbers are already on the board at each stage, so no c is to be written down, unless $n = 4k - 1$ in which case $4k$ is written down during the first move. The outcome of this k-move round is the string of even numbers 2 through $4k$ written down on the board. At this stage, the problem is clearly brought down to the case where the numbers on the board are 1, 2, …, $2k = \lceil n/2 \rceil$, as desired.

Finally, let $n = 4k + 1$ or $4k + 2$, where $k \geq 2$. Remove first the pair $(4, 2k + 1)$ and notice that no new number is to be written down on the board, since $4 + (2k + 1) = 2k + 5 \leq 4k + 1 \leq n$. Next, remove the pairs $(1, 4k + 1)$, $(3, 4k - 1)$, …, $(2k - 1, 2k + 3)$ in turn. As before, at each of these stages, two odd numbers are removed; the corresponding $c = |a \pm b|$ are even numbers, this time in the range 4 through $4k + 2$, of which one is always $4k + 2$; and no new numbers are to be written down on the board, except $4 = |(2k - 1) - (2k + 3)|$ during the last move, and, possibly, $4k + 2 = 1 + (4k + 1)$ during the first move if $n = 4k + 1$. Notice that 2 has not yet been involved in the process, to conclude that the outcome of this $(k + 1)$-move round is the string of even numbers 2 through $4k + 2$ written down on the board. At this stage, the problem is clearly brought down to the case where the numbers on the board are 1, 2, …, $2k + 1 = \lceil n/2 \rceil$, as desired.

Solution 2. We will prove the following, more general statement:

Claim. Write down a finite number (at least two) of pairwise distinct positive integers on a board. A *move* consists in erasing any two numbers a and b, and, for each c in $\{a + b, |a - b|\}$, writing c down on the board, unless c is already there; if c is already on the board, do nothing. Then it is possible to be left with exactly two numbers on the board after a finite number of moves.

Notice that, if we divide all numbers on the board by some common factor, the resulting process goes on equally well. Such a reduction can therefore be performed after any move.

Notice that we cannot be left with less than two numbers. So it suffices to show that, given k positive integers on the board, $k \geq 3$, we can always decrease their number by at least 1. Arguing indirectly, choose a set of $k \geq 3$ positive integers $S = \{a_1, \ldots, a_k\}$ which cannot be reduced in size by a sequence of moves, having a minimal possible sum σ. So, in any sequence of moves applied to S, two numbers are erased and exactly two numbers appear on each move. Moreover, the sum of any resulting set of k numbers is at least σ.

Notice that, given two numbers $a > b$ on the board, we can replace them by $a + b$ and $a - b$, and then, performing a move on the two new numbers, by $(a + b) + (a - b) = 2a$ and $(a + b) - (a - b) = 2b$. So we can double any two numbers on the board.

We now show that, if the board contains two even numbers a and b, we can divide them both by 2, while keeping the other numbers unchanged. If k is even, split the other numbers into pairs to multiply each pair by 2; then clear out the common factor 2. If k is odd, split all numbers but a into pairs to multiply each by 2; then do the same for all numbers but b; finally, clear out the common factor 4.

Back to the problem, if two of the numbers $a_1, \ldots, a_k$ are even, reduce them both by 2 to get a set with a smaller sum, which is impossible. Otherwise, two numbers, say, $a_1 < a_2$, are odd, and we may replace them by the two even numbers $a_1 + a_2$ and $a_2 - a_1$, and then by $\frac{1}{2}(a_1 + a_2)$ and $\frac{1}{2}(a_2 - a_1)$, to get a set with a smaller sum, which is again impossible.

Problem 5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and $60°$ rotational symmetry (that is, there is a point O such that a $60°$ rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2+3n+1$ citizens at $3n^2+3n+1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

USA, Ankan Bhattacharya

Solution. Let P denote the given polygon, i.e., the kingdom of Zoomtopia. Throughout the solution, we interpret polygons with integer sides and perimeter $6k$ as $6k$-gons with unit sides (some of their angles may equal $180°$). The argument hinges on the claim below:

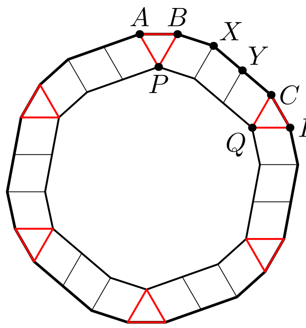
Claim. Let P be a convex polygon satisfying the problem conditions — i.e., it has integer sides, perimeter $6n$, and $60°$ rotational symmetry. Then P can be tiled with unit equilateral triangles and unit lozenges with angles at least $60°$, with tiles meeting completely along edges, so that the tile configuration has a total of exactly $3n^2+3n+1$ distinct vertices.

Proof. Induct on n. The base case, $n=1$, is clear.

Now take a polygon P of perimeter $6n \geq 12$. Place six equilateral triangles inwards on six edges corresponding to each other upon rotation at $60°$. It is possible to stick a lozenge to each other edge, as shown in the Figure below.

We show that all angles of the lozenges are at least $60°$. Let an edge XY of the polygon bearing some lozenge lie along a boundary segment between edges AB and CD bearing equilateral triangles ABP and CDQ. Then the angle formed by $\overrightarrow{XY}$ and $\overrightarrow{BP}$ is between those formed by $\overrightarrow{AB}$, $\overrightarrow{BP}$ and $\overrightarrow{CD}$, $\overrightarrow{CQ}$, i.e., between $60°$ and $120°$, as desired.

Removing all obtained tiles, we get a $60°$-symmetric convex $6(n-1)$-gon with unit sides which can be tiled by the inductive hypothesis. Finally, the number of vertices in the tiling of P is $6n+3(n-1)^2+3(n-1)+1 = 3n^2+3n+1$, as desired.



Using the Claim above, we now show that the citizens may be placed at the $3n^2+3n+1$ tile vertices.

Consider any tile T_1; its vertices are at least 1 apart from each other. Moreover, let BAC be a part of the boundary of some tile T, and let X be any point of the boundary of T, lying outside the half-open intervals $[A,B)$ and $[A,C)$ (in this case, we say that X is not adjacent to A). Then $AX \geq \sqrt{3}/2$.

Now consider any two tile vertices A and B. If they are vertices of the same tile we already know $AB \geq 1$; otherwise, the segment AB crosses the boundaries of some tiles containing A and B at some points X and Y not adjacent to A and B, respectively. Hence $AB \geq AX+YB \geq \sqrt{3} > 1$.

Problem 6. Initially, a non-constant polynomial $S(x)$ with real coefficients is written down on a board. Whenever the board contains a polynomial $P(x)$, not necessarily alone, one can write down on the board any polynomial of the form $P(C+x)$ or $C+P(x)$, where C is a real constant. Moreover, if the board contains two (not necessarily distinct) polynomials $P(x)$ and $Q(x)$, one can write $P(Q(x))$ and $P(x)+Q(x)$ down on the board. No polynomial is ever erased from the board.

Given two sets of real numbers, $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$, a polynomial $f(x)$ with real coefficients is (A,B)-*nice* if $f(A) = B$, where $f(A) = \{f(a_i)\colon i = 1, 2, \dots, n\}$.

Determine all polynomials $S(x)$ that can initially be written down on the board such that, for any two finite sets A and B of real numbers, with $|A| = |B|$, one can produce an (A,B)-nice polynomial in a finite number of steps.

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Solution. The required polynomials are all polynomials of an even degree $d \geq 2$, and all polynomials of odd degree $d \geq 3$ with negative leading coefficient.

Part I. We begin by showing that any (non-constant) polynomial $S(x)$ **not** listed above is not (A,B)-nice for some pair (A,B) with either $|A| = |B| = 2$, or $|A| = |B| = 3$.

If $S(x)$ is linear, then so are all the polynomials appearing on the board. Therefore, none of them will be (A,B)-nice, say, for $A = \{1,2,3\}$ and $B = \{1,2,4\}$, as desired.

Otherwise, $\deg S = d \geq 3$ is odd, and the leading coefficient is positive. In this case, we make use of the following technical fact, whose proof is presented at the end of the solution.

Claim. There exists a positive constant T such that $S(x)$ satisfies the following condition:

$$S(b) - S(a) \geq b - a \qquad \text{whenever} \quad b - a \geq T. \tag{$*$}$$

Fix a constant T provided by the Claim. Then, an immediate check shows that all newly appearing polynomials on the board also satisfy $(*)$ (with the same value of T). Therefore, none of them will be (A,B)-nice, say, for $A = \{0, T\}$ and $B = \{0, T/2\}$, as desired.

Part II. We show that the polynomials listed in the Answer satisfy the requirements. We will show that for any $a_1 < a_2 < \dots < a_n$ and any $b_1 \leq b_2 \leq \dots \leq b_n$ there exists a polynomial $f(x)$ satisfying $f(a_i) = b_{\sigma(i)}$ for all $i = 1, 2, \dots, n$, where σ is some permutation.

The proof goes by induction on $n \geq 2$. It is based on the following two lemmas, first of which is merely the base case $n = 2$; the proofs of the lemmas are also at the end of the solution.

Lemma 1. For any $a_1 < a_2$ and any b_1, b_2 one can write down on the board a polynomial $F(x)$ satisfying $F(a_i) = b_i$, $i = 1, 2$.

Lemma 2. For any distinct numbers $a_1 < a_2 < \dots < a_n$ one can produce a polynomial $F(x)$ on the board such that the list $F(a_1)$, $F(a_2)$, $\dots$, $F(a_n)$ contains exactly $n-1$ distinct numbers, and $F(a_1) = F(a_2)$.

Now, in order to perform the inductive step, we may replace the polynomial $S(x)$ with its shifted copy $S(C+x)$ so that the values $S(a_i)$ are pairwise distinct. Applying Lemma 2, we get a polynomial $f(x)$ such that only two among the numbers $c_i = f(a_i)$ coincide, namely c_1 and c_2. Now apply Lemma 1 to get a polynomial $g(x)$ such that $g(a_1) = b_1$ and $g(a_2) = b_2$. Apply the inductive hypothesis in order to obtain a polynomial $h(x)$ satisfying $h(c_i) = b_i - g(a_i)$ for all $i = 2, 3, \dots, n$. Then the polynomial $h(f(x)) + g(x)$ is a desired one; indeed, we have $h(f(a_i)) + g(a_i) = h(c_i) + g(a_i) = b_i$ for all $i = 2, 3, \dots, n$, and finally $h(f(a_1)) + g(a_1) = h(c_1) + g(a_1) = b_2 - g(a_2) + g(a_1) = b_1$.

It remains to prove the Claim and the two Lemmas.

Proof of the Claim. There exists some segment $\Delta = [\alpha', \beta']$ such that $S(x)$ is monotone increasing outside that segment. Now one can choose $\alpha \leq \alpha'$ and $\beta \geq \beta'$ such that $S(\alpha) < \min_{x \in \Delta} S(x)$ and

$S(\beta) > \max_{x\in\Delta} S(x)$. Therefore, for any x, y, z with $x \le \alpha \le y \le \beta \le z$ we get $S(x) \le S(\alpha) \le S(y) \le S(\beta) \le S(z)$.

We may decrease α and increase β (preserving the condition above) so that, in addition, $S'(x) > 3$ for all $x \notin [\alpha, \beta]$. Now we claim that the number $T = 3(\beta - \alpha)$ fits the bill.

Indeed, take any a and b with $b - a \ge T$. Even if the segment $[a, b]$ crosses $[\alpha, \beta]$, there still is a segment $[a', b'] \subseteq [a, b] \setminus (\alpha, \beta)$ of length $b' - a' \ge (b - a)/3$. Then

$$S(b) - S(a) \ge S(b') - S(a') = (b' - a') \cdot S'(\xi) \ge 3(b' - a') \ge b - a$$

for some $\xi \in (a', b')$.

Proof of Lemma 1. If $S(x)$ has an even degree, then the polynomial $T(x) = S(x+a_2) - S(x+a_1)$ has an odd degree, hence there exists x_0 with $T(x_0) = S(x_0+a_2) - S(x_0+a_1) = b_2 - b_1$. Setting $G(x) = S(x+x_0)$, we see that $G(a_2) - G(a_1) = b_2 - b_1$, so a suitable shift $F(x) = G(x) + \big(b_1 - G(a_1)\big)$ fits the bill.

Assume now that $S(x)$ has odd degree and a negative leading coefficient. Notice that the polynomial $S^2(x) := S(S(x))$ has an odd degree and a positive leading coefficient. So, the polynomial $S^2(x+a_2) - S^2(x+a_1)$ attains all sufficiently large positive values, while $S(x+a_2) - S(x+a_1)$ attains all sufficiently large negative values. Therefore, the two-variable polynomial $S^2(x+a_2) - S^2(x+a_1) + S(y+a_2) - S(y+a_1)$ attains all real values; in particular, there exist x_0 and y_0 with $S^2(x_0+a_2) + S(y_0+a_2) - S^2(x_0+a_1) - S(y_0+a_1) = b_2 - b_1$. Setting $G(x) = S^2(x+x_0) + S(x+y_0)$, we see that $G(a_2) - G(a_1) = b_2 - b_1$, so a suitable shift of G fits the bill.

Proof of Lemma 2. Let Δ denote the segment $[a_1; a_n]$. We modify the proof of Lemma 1 in order to obtain a polynomial F convex (or concave) on Δ such that $F(a_1) = F(a_2)$; then F is a desired polynomial. Say that a polynomial $H(x)$ is *good* if H is convex on Δ.

If $\deg S$ is even, and its leading coefficient is positive, then $S(x+c)$ is good for all sufficiently large negative c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large negative values for such c. Similarly, $S(x+c)$ is good for all sufficiently large positive c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large positive values for such c. Therefore, there exist large $c_1 < 0 < c_2$ such that $S(x+c_1) + S(x+c_2)$ is a desired polynomial. If the leading coefficient of H is negative, we similarly find a desired polynomial which is concave on Δ.

If $\deg S \ge 3$ is odd (and the leading coefficient is negative), then $S(x+c)$ is good for all sufficiently large negative c, and $S(a_2+c) - S(a_1+c)$ attains all sufficiently large negative values for such c. Similarly, $S^2(x+c)$ is good for all sufficiently large positive c, and $S^2(a_2+c) - S^2(a_1+c)$ attains all sufficiently large positive values for such c. Therefore, there exist large $c_1 < 0 < c_2$ such that $S(x+c_1) + S^2(x+c_2)$ is a desired polynomial.

Comment. Both parts above allow some variations.

In Part I, the same scheme of the proof works for many conditions similar to $(*)$, e.g.,

$$S(b) - S(a) > T \qquad \text{whenever} \quad b - a > T.$$

Let us sketch an alternative approach for Part II. It suffices to construct, for each i, a polynomial $f_i(x)$ such that $f_i(a_i) = b_i$ and $f_i(a_j) = 0$, $j \ne i$. The construction of such polynomials may be reduced to the construction of those for $n = 3$ similarly to what happens in the proof of Lemma 2. However, this approach (as well as any in this part) needs some care in order to work properly.