# The 15$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2: Thursday, February 29$^{\text{th}}$, 2024, Bucharest

Language: English

**Problem 4.** Fix integers $a$ and $b$ greater than 1. For any positive integer $n$, let $r_n$ be the (non-negative) remainder that $b^n$ leaves upon division by $a^n$. Assume there exists a positive integer $N$ such that $r_n < 2^n/n$ for all integers $n \geq N$. Prove that $a$ divides $b$.

**Problem 5.** Let $BC$ be a fixed segment in the plane, and let $A$ be a variable point in the plane not on the line $BC$. Distinct points $X$ and $Y$ are chosen on the rays $\overrightarrow{CA}$ and $\overrightarrow{BA}$, respectively, such that $\angle CBX = \angle YCB = \angle BAC$. Assume that the tangents to the circumcircle of $ABC$ at $B$ and $C$ meet line $XY$ at $P$ and $Q$, respectively, such that the points $X$, $P$, $Y$, and $Q$ are pairwise distinct and lie on the same side of $BC$. Let $\Omega_1$ be the circle through $X$ and $P$ centred on $BC$. Similarly, let $\Omega_2$ be the circle through $Y$ and $Q$ centred on $BC$. Prove that $\Omega_1$ and $\Omega_2$ intersect at two fixed points as $A$ varies.

**Problem 6.** A polynomial $P$ with integer coefficients is *square-free* if it is not expressible in the form $P = Q^2R$, where $Q$ and $R$ are polynomials with integer coefficients and $Q$ is not constant. For a positive integer $n$, let $\mathcal{P}_n$ be the set of polynomials of the form

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n$$

with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. Prove that there exists an integer $N$ so that, for all integers $n \geq N$, more than 99% of the polynomials in $\mathcal{P}_n$ are square-free.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.